SIMPLE LIE ALGEBRAS OF SPECIAL TYPE

Jeffrey Bergen
D. S. Passman

DePaul University
University of Wisconsin-Madison

Abstract. Let K be a field, let A be an associative, commutative K-algebra and let Δ be a nonzero K-vector space of commuting K-derivations of A. Then, with a rather natural definition, $\mathbb{W}(A,\Delta) = A\otimes_K \Delta = A\Delta$ becomes a Lie algebra, a Witt type algebra. In addition, there is a map $\mathrm{div}\colon \mathbb{W}(A,\Delta)\to A$ called the divergence and its kernel $S = \mathbb{S}(A,\Delta)$ is a Lie subalgebra, a special type algebra. In this paper, we study S from a ring theoretic point of view, obtaining sufficient conditions for the Lie simplicity of $[S,S]$. While the main result here is somewhat cumbersome to state, it does handle a number of examples in a fairly efficient manner. Furthermore, some of the preliminary lemmas are of interest in their own right and may, in time, lead to a more satisfactory answer.

§1. Introduction

Let K be a field, let A be an associative, commutative K-algebra and let Δ be a nonzero K-vector space of commuting K-derivations of A. Then the tensor product $A\otimes_K \Delta = A\Delta$ acts on A by way of

$$a\otimes\partial\colon x\mapsto a\partial(x)\qquad \text{for all } a,x\in A,\ \partial\in\Delta.$$

Since A is commutative, this gives rise to a linear transformation

$$\theta\colon A\Delta\to \mathrm{Der}_K(A)\subseteq \mathrm{Hom}_K(A,A).$$

Furthermore, suppose $a,b\in A$ and $\alpha,\beta\in\Delta$. Then, since α and β commute, we obtain the equality

$$a\alpha{\cdot}b\beta - b\beta{\cdot}a\alpha = a\alpha(b)\beta - b\beta(a)\alpha$$

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as operators on A. Consequently, the image of θ is a Lie subalgebra of $\operatorname{Der}_K(A)$. Indeed, the preceding displayed equation motivates the definition of the binary operation $[\ ,\ ]$ on $A\Delta$ as the K-linear extension of

$$[a\alpha, b\beta] = a\alpha(b)\beta - b\beta(a)\alpha \qquad \text{for all } a, b \in A,\ \alpha, \beta \in \Delta.$$

As is well known, this yields a Lie algebra structure on $A\Delta$ and then θ is clearly a Lie homomorphism. We denote this Witt type Lie algebra by $W = \mathbb{W}(A, \Delta)$.

The standard example of this construction is the Witt algebra $W_n = A\Delta$. Here $A = K[t_1^{\pm}, t_2^{\pm}, \ldots, t_n^{\pm}]$ is the ring of Laurent polynomials in the variables $t_1, t_2, \ldots, t_n$, and Δ is the K-vector space spanned by the partial derivatives $\partial/\partial t_i$. More general versions of this construction have been considered by Kaplansky [Kp], Kawamoto [Kw], Osborn [O], Đoković and Zhao [DZ], Passman [P] and others.

Now let us define the divergence map $\operatorname{div}: W = A\Delta \to A$ to be the K-linear extension of

$$\operatorname{div}(a\alpha) = \alpha(a) \qquad \text{for all } a \in A,\ \alpha \in \Delta.$$

Basic properties are as follows.

Lemma 1.1. *With the above notation, we have*

i. $\operatorname{div}([v, w]) = v(\operatorname{div}(w)) - w(\operatorname{div}(v))$ *for all* $v, w \in A\Delta$.
ii. $S = \{w \in A\Delta \mid \operatorname{div}(w) = 0\}$ *is a Lie subalgebra of* $A\Delta$.

Proof. Since the divergence map is K-linear, it suffices to check (i) on the elements $v = a\alpha$ and $w = b\beta$ for all $a, b \in A$ and $\alpha, \beta \in \Delta$. Here, we have

$$\begin{aligned}\operatorname{div}([v, w]) = \operatorname{div}([a\alpha, b\beta]) &= \beta(a\alpha(b)) - \alpha(b\beta(a)) = a\cdot\beta\alpha(b) - b\cdot\alpha\beta(a)\\ &= (a\alpha)(\beta(b)) - (b\beta)(\alpha(a)) = v(\operatorname{div}(w)) - w(\operatorname{div}(v)),\end{aligned}$$

as required. Part (ii) is, of course, an immediate consequence of (i). □

We denote the kernel of this divergence map by $S = \mathbb{S}(A, \Delta)$. The goal here is to discuss the Lie simplicity of S and, more precisely, of its commutator ideal $[S, S]$.

Let $\mathcal{B} = \{\partial_i \mid i \in \mathcal{I}\}$ be a fixed K-basis for Δ, and let the map $D_{i,j}: A \to A\Delta$ be defined by

$$D_{i,j}(a) = D_{\partial_i, \partial_j}(a) = \partial_i(a)\partial_j - \partial_j(a)\partial_i$$

for all $i, j \in \mathcal{I}$. Note that $D_{i,i} = 0$ and that $D_{j,i} = -D_{i,j}$.

Lemma 1.2. *If S and $D_{i,j}$ are given as above, then*

i. $D_{i,j}$ *is a K-linear map into S.*
ii. *If* $\sum_i a_i\partial_i \in S$, *then* $(\sum_i a_i\partial_i)(b) = \sum_i \partial_i(a_i b)$ *for all* $b \in A$.
iii. $[\sum_i a_i\partial_i, \sum_j b_j\partial_j] = \sum_{i,j} D_{i,j}(a_i b_j)$ *provided* $\sum_i a_i\partial_i,\ \sum_j b_j\partial_j \in S$.
iv. $[1\partial, D_{i,j}(a)] = D_{i,j}(\partial(a))$ *for all* $a \in A$ *and* $\partial \in \Delta$.

Proof. Parts (i) and (iv) are clear. For (ii), observe that if $s = \sum_i a_i\partial_i \in S$ then $\sum_i \partial_i(a_i)b = b{\cdot}\mathrm{div}(s) = 0$, and hence

$$\Big(\sum_i a_i\partial_i\Big)(b) = \sum_i (a_i\partial_i(b) + \partial_i(a_i)b) = \sum_i \partial_i(a_i b).$$

Finally, (ii) yields

$$\begin{aligned}\Big[\sum_i a_i\partial_i, \sum_j b_j\partial_j\Big] &= \sum_j \Big(\sum_i a_i\partial_i\Big)(b_j){\cdot}\partial_j - \sum_i\Big(\sum_j b_j\partial_j\Big)(a_i){\cdot}\partial_i \\ &= \sum_{i,j}(\partial_i(a_ib_j)\partial_j - \partial_j(a_ib_j)\partial_i) = \sum_{i,j} D_{i,j}(a_ib_j),\end{aligned}$$

and (iii) is proved. □

If we define $D = \mathbb{D}(A,\Delta)$ to be the K-linear span of all $D_{i,j}(a)$ with $i,j \in \mathcal{I}$ and $a \in A$, then (i) and (iii) above show that $S \supseteq D \supseteq [S,S]$. Thus D is a Lie ideal of S which is easily seen to be independent of the choice of basis.

In the remainder of this paper, the elements $i,j,k \in \mathcal{I}$ are always assumed to be distinct. Indeed, whenever we write i,j or k, then they are assumed to be in the index set $\mathcal{I}$. Furthermore, we let

$$A^i = A^{\partial_i} = \{a \in A \mid \partial_i(a) = 0\},$$

$A^{i,j} = A^i \cap A^j$, and if $\mathcal{I}'$ is a subset of $\mathcal{I}$, then

$$A^{\mathcal{I}'} = \bigcap_{i\in\mathcal{I}'} A^i.$$

If $\dim_L \Delta = 1$, then $S = A^1\partial_1$ is abelian and $D = 0$. Thus, it suffices to assume throughout that $\dim_K \Delta \geq 2$. Here we have

Lemma 1.3. *If* $\dim_K \Delta \geq 2$*, then*

- i. $\Delta(A^i)\partial_i \subseteq D$ *for all* $i \in \mathcal{I}$.
- ii. $D_{i,j}(\Delta(A^i){\cdot}\Delta(A^j)) \subseteq [D,D]$ *for all* $i,j \in \mathcal{I}$.
- iii. *If for some* $i \in \mathcal{I}$ *we have* $\partial_i(A) = A$ *and* $\Delta(A^i) = A^i$*, then* $S = D$.

Proof. (i) If $a \in A^i$, then $D_{j,i}(a) = \partial_j(a)\partial_i - \partial_i(a)\partial_j = \partial_j(a)\partial_i$. Thus $\partial_j(A^i)\partial_i = D_{j,i}(A^i) \subseteq D$ and, since $\partial_i(A^i)\partial_i = 0 \subseteq D$, we conclude that $\Delta(A^i)\partial_i \subseteq D$.

(ii) This follows from (i) and Lemma 1.2(iii) since $[D,D] \supseteq [\Delta(A^i)\partial_i, \Delta(A^j)\partial_j] = D_{i,j}(\Delta(A^i){\cdot}\Delta(A^j))$.

(iii) We may assume that $\partial_1(A) = A$ and $\Delta(A^1) = A^1$. Let $s = \sum_1^n a_i\partial_i \in S$. Since ∂_1 is onto, choose $x_i \in A$ with $\partial_1(x_i) = a_i$. Then

$$D_{1,i}(x_i) = \partial_1(x_i)\partial_i - \partial_i(x_i)\partial_1 = a_i\partial_i - \partial_i(x_i)\partial_1,$$

so it follows that $s - \sum_2^n D_{1,i}(x_i) \in S \cap A\partial_1 = A^1\partial_1$. But $S \cap A\partial_1 = A^1\partial_1 = \Delta(A^1)\partial_1$, by assumption, and $\Delta(A^1)\partial_1 \subseteq D$ by (i). Thus we conclude that $s \in D$ and hence that $S = D$. □

The goal of this paper is to discuss the Lie simplicity of S and, more precisely, of $[S, S]$. Suppose that S itself is Lie simple. If I is a nonzero Δ-stable associative ideal of A, then $I\Delta$ is a Lie ideal of $W = A\Delta$ and hence $I\Delta \cap S$ is a Lie ideal of S. Furthermore, $I\Delta \cap S \neq 0$ since if $0 \neq a \in I$, then either $\partial_1(a) = 0$ and $0 \neq a\partial_1 \in I\Delta \cap S$ or $\partial_1(a) \neq 0$ and $0 \neq D_{1,2}(a) = \partial_1(a)\partial_2 - \partial_2(a)\partial_1 \in I\Delta \cap S$. Thus, since S is Lie simple, we must have $I\Delta \supseteq S$ and, since $1\partial_1 \in S$, we conclude that $1 \in I$. In other words, $I = A$ and A is Δ-simple. In this case, A and the set

$$A^\Delta = A^\mathcal{I} = \{a \in A \mid \partial(a) = 0 \text{ for all } \partial \in \Delta\}$$

of common constants have some rather nice properties.

Lemma 1.4. *Assume that A is Δ-simple. Then*

i. *A^Δ is a field containing K.*
ii. *Any nonzero Δ-stable subset of A is regular, that is not a zero divisor in A.*
iii. *If $\operatorname{char} K = 0$, then A is a domain.*
iv. *If $\operatorname{char} K = p > 0$, then $a^p \in A^\Delta$ for all $a \in A$. In particular, if N is the nil radical of A, then A/N is a field.*

Proof. We know that A^Δ is a subring of A. Furthermore, if $0 \neq a \in A^\Delta$, then aA is a nonzero Δ-stable ideal of A and hence $aA = A$. Thus a is invertible in A and, since $\partial(1/a) = -\partial(a)/a^2 = 0$, it follows that $1/a \in A^\Delta$. Consequently (i) is proved, and (ii) is obvious. Furthermore, (iii) is a standard result (see for example [McR, Proposition 14.2.4]). Finally, suppose $\operatorname{char} K = p > 0$. Then certainly $a^p \in A^\Delta$ for all $a \in A$. In particular, (i) implies that a is either nilpotent or invertible, and this clearly yields (iv). □

Now, if $\partial \in \Delta$, then we know that 1∂ need not be contained in D. Because of this, it is usually not obvious that certain subsets of A are Δ-stable. One technique we use to overcome this difficulty is based on the following computation.

Lemma 1.5. *Let $\partial \in \Delta$ and let $\mathcal{T}_\partial$ be the linear operator on $W = A\Delta$ defined by $\mathcal{T}_\partial(a\alpha) = \partial(a)\alpha$ for all $a \in A$ and $\alpha \in \Delta$. If X is a K-subspace of A^∂ with $1 \in X^n$ some some $n \geq 1$, then $(\operatorname{ad}_{X\partial})^n$ contains the operator $\mathcal{T}_\partial^n$.*

Proof. We first show, by induction on $m \geq 1$, that if $x_1, x_2, \ldots, x_m \in X$, then

$$\operatorname{ad}_{x_1\partial} \operatorname{ad}_{x_2\partial} \cdots \operatorname{ad}_{x_m\partial} \colon a\alpha \mapsto x_1x_2\cdots x_m\partial^m(a)\alpha - \alpha(x_1x_2\cdots x_m)\partial^{m-1}(a)\partial.$$

To start with, we have $\operatorname{ad}_{x_1\partial}(a\alpha) = [x_1\partial, a\alpha] = x_1\partial(a)\alpha - a\alpha(x_1)\partial$, so the $m = 1$ result holds. Now suppose that $m \geq 2$ and that the $m - 1$ case is satisfied. Set

$y = x_2 \cdots x_m$, and observe that y, $\alpha(y)$ and x_1 are all contained in A^∂. Then, by induction, we have

$$\begin{aligned}
\mathrm{ad}_{x_1\partial}\,\mathrm{ad}_{x_2\partial}\cdots\mathrm{ad}_{x_m\partial}(a\alpha) &= \mathrm{ad}_{x_1\partial}\big(y\partial^{m-1}(a)\alpha - \alpha(y)\partial^{m-2}(a)\partial\big) \\
&= [x_1\partial, y\partial^{m-1}(a)\alpha - \alpha(y)\partial^{m-2}(a)\partial] \\
&= x_1\partial\big(y\partial^{m-1}(a)\big)\alpha - x_1\partial\big(\alpha(y)\partial^{m-2}(a)\big)\partial \\
&\qquad - y\partial^{m-1}(a)\alpha(x_1)\partial + \alpha(y)\partial^{m-2}(a)\partial(x_1)\partial \\
&= x_1 y\partial^m(a)\alpha - \big(x_1\alpha(y) + \alpha(x_1)y\big)\partial^{m-1}(a)\partial \\
&= x_1 y\partial^m(a)\alpha - \alpha(x_1 y)\partial^{m-1}(a)\partial,
\end{aligned}$$

as required. By linearity, it now follows that $(\mathrm{ad}_{X\partial})^n$ contains the maps

$$a\alpha \mapsto s\partial^n(a)\alpha - \alpha(s)\partial^{n-1}(a)\partial$$

for all $s \in X^n$. In particular, if $1 \in X^n$ then, since $\alpha(1) = 1$, we see that $(\mathrm{ad}_{X\partial})^n$ contains the map $\mathcal{T}^n_\partial\colon a\alpha \mapsto \partial^n(a)\alpha$. $\square$

Again, suppose that S is Lie simple. Since $S \supseteq 1\Delta$ and 1Δ acts faithfully on A, it follows that S must also act faithfully on A. In particular, since $S \supseteq A^\Delta\Delta$, we see that $A^\Delta\Delta$ is faithful on A. Thus, in studying the simplicity of S, it is reasonable to assume that A is Δ-simple and that $A^\Delta\Delta$ acts faithfully on A. On the other hand, if $[S,S]$ is Lie simple, then it is not at all clear that these conditions must be satisfied. Nevertheless, we assume them in the main result of this paper.

Theorem 1.6. *Let K be a field, let A be a commutative K-algebra, and let Δ be a K-subspace of $\mathrm{Der}_K(A)$ of dimension ≥ 2 consisting of commuting derivations. Write $S = \mathbb{S}(A,\Delta)$, $D = \mathbb{D}(A,\Delta)$, and let $\mathcal{B} = \{\partial_i \mid i \in \mathcal{I}\}$ be a fixed K-basis for Δ. Assume that*

1. *A is Δ-simple and $A^\Delta\Delta$ acts faithfully on A.*
2. *$\partial_i(A^j) \neq 0$ for all i, j, and $\partial_i(A^{j,k}) \neq 0$ for all i, j, k.*
3. *Either $\mathrm{char}\, K \neq 2$ or $\partial_i(\Delta(A^j)) \neq 0$ for all distinct i, j.*
4. *One of the following two conditions is satisfied.*
   - (a) *$\Delta(A^i) \cap A^{\mathcal{I}'} \neq 0$ for all finite subsets $\mathcal{I}'$ of the index set $\mathcal{I}$.*
   - (b) *Each ∂_i is diagonalizable on A, and either $\mathrm{char}\, K = p > 0$ or $1 \in \Delta(A^i)^n$ for at least two relatively prime integers n.*
5. *One of the following two conditions is satisfied.*
   - (a) *$\partial_i(A) + \partial_j(A) + A^{i,j} = A$ for all distinct i, j.*
   - (b) *$1 \in \partial_i\big(\partial_j(A) \cap A^j\big)$, and either $A = \Delta(A) + A^{i,j}$ or $\dim_K \Delta \geq 3$ and $\big(\partial_i(A^j) \cap A^i\big)\cdot\partial_k(A) = A$ for all i, j, k.*

Then $D = [S,S]$ is Lie simple.

Note that the assumption $\partial_i(A^j) \neq 0$ in (2) follows from $\partial_i(A^{j,k}) \neq 0$ provided that k exists, that is provided $\dim_K \Delta \geq 3$. Furthermore, since the five conditions in

Theorem 1.6 are certainly not necessary for the simplicity of D, it is not surprising that alternate hypotheses exist which can yield the same conclusion. In particular, we are aware of several replacements for conditions (4) and (5). However, as we have indicated, this result is far from definitive, and therefore we have chosen to include only those conditions required to handle the examples discussed in §3. We suspect that, in the long run, the arguments and lemmas of the next section will prove to be of more interest than the theorem itself.

In closing, the authors would like to thank Georgia Benkart for suggesting this problem and for supplying the basic formulas in [B]. The second author would also like to point out that the char $K \neq 2$ work in [P] follows from results of D. A. Jordan in [J1] and [J2] once one knows that the Witt type algebra W acts faithfully on A.

§2. Poisson Brackets

In this section, we prove the main result, introducing each assumption at the point it is required. Recall that a map $\{\ ,\ \}\colon A\times A \to A$ is said to be a Poisson bracket for A if A becomes a K-Lie algebra under $\{\ ,\ \}$ and if the maps $\mathrm{ad}_a\colon A \to A$ defined by $\mathrm{ad}_a(x) = \{a, x\}$ are derivations of the associative algebra A. In particular, the latter says that

$$\{a, bc\} = \{a, b\}c + \{a, c\}b \qquad \text{for all } a, b, c \in A.$$

Furthermore, interchanging a and b in the above yields $\{b, ac\} = \{b, a\}c + \{b, c\}a$, and by adding this to the original formula we obtain

$$\{a, bc\} + \{b, ac\} = \{a, c\}b + \{b, c\}a = \{ab, c\} \qquad \text{for all } a, b, c \in A$$

since $\{a, b\} = -\{b, a\}$. A key observation is as follows.

Lemma 2.1. *Let A be a commutative algebra with Poisson bracket $\{\ ,\ \}$ and let L be a Lie ideal of A. Then $L \supseteq \{A, I\}$, where I is the associative ideal $I = \{L, L\}\cdot A$. Furthermore, if $\{L, L\} = 0$ and $\mathrm{char}\, K \neq 2$, then $J = \{L, A\}\cdot A$ is a nil ideal of A.*

Proof. In the first displayed equation above, let $a, c \in L$ and $b \in A$. Then $\{a, bc\} \in L$ and $\{a, b\}c \in L^2$. Thus $\{a, c\}b \in L + L^2$ and consequently $L + L^2$ contains the ideal $I = \{L, L\}\cdot A$.

Next, in the second displayed equation, let $a, b \in L$ and $c \in A$. Then $\{a, bc\}$ and $\{b, ac\}$ are both in L, so $\{c, ab\} = -\{ab, c\} \in L$. In other words, $\{A, L^2\} \subseteq L$ and we conclude that $L \supseteq \{A, L + L^2\} \supseteq \{A, I\}$.

Finally, suppose $\{L, L\} = 0$. If $a \in L$ and $b \in A$, then $\mathrm{ad}_a(b) \in L$, so $\mathrm{ad}_a^2(b) \in \{L, L\} = 0$. Thus,

$$0 = \mathrm{ad}_a^2(b^2) = 2\,\mathrm{ad}_a(b\cdot\mathrm{ad}_a(b)) = 2\,\mathrm{ad}_a(b)^2 + 2\,b\cdot\mathrm{ad}_a^2(b) = 2\,\mathrm{ad}_a(b)^2,$$

and hence $\mathrm{ad}_a(b)^2 = 0$ if $\mathrm{char}\, K \neq 2$. In particular, under this assumption on the characteristic of K, $J = \{L, A\}{\cdot}A$ is generated by the nilpotent elements $\{a, b\} = \mathrm{ad}_a(b)$, and consequently it is a nil ideal. □

Now let Δ be a K-vector space of commuting derivations of A and, as usual, let $\mathcal{B} = \{\partial_i \mid i \in \mathcal{I}\}$ be a fixed K-basis for Δ. We assume throughout this section that A is Δ-simple and that $A^\Delta\Delta$ acts faithfully on A. By [P, Proposition 2.4], this implies that $W = A\Delta$ acts faithfully on A. For each distinct i, j in the index set $\mathcal{I}$, define the Jacobian map $\{\ ,\ \}_{i,j}\colon A \times A \to A$ by

$$\{a, b\}_{i,j} = \partial_i(a)\partial_j(b) - \partial_j(a)\partial_i(b).$$

Then we have

Lemma 2.2. *Let $i, j \in \mathcal{I}$.*

i. *$\{\ ,\ \}_{i,j}$ is a Poisson bracket for A and Δ acts as derivations on the Lie algebra $A, \{\ ,\ \}_{i,j}$.*
ii. *The map $\theta_{i,j}\colon a \mapsto D_{i,j}(a)$ is a Lie epimorphism from A with the Poisson bracket $\{\ ,\ \}_{i,j}$ to $D_{i,j}(A)$.*
iii. *$D_{i,j}(a)(b) = \{a, b\}_{i,j}$ for all $a, b \in A$ and hence the kernel of $\theta_{i,j}$ is $A^{i,j}$, the center of the Lie algebra $A, \{\ ,\ \}_{i,j}$.*

Proof. (i) It is well-known and easily verified that $\{\ ,\ \}_{i,j}$ is a Poisson bracket for A. Furthermore, since all $\partial \in \Delta$ commute with ∂_i and ∂_j, it is clear from the definition that $\partial\big(\{a, b\}_{i,j}\big) = \{\partial(a), b\}_{i,j} + \{a, \partial(b)\}_{i,j}$.

(ii) $\theta_{i,j}$ is clearly K-linear. In addition, by Lemma 1.2(iii), we have

$$\begin{aligned}[D_{i,j}(a), D_{i,j}(b)] &= [\partial_i(a)\partial_j - \partial_j(a)\partial_i, \partial_i(b)\partial_j - \partial_j(b)\partial_i] \\ &= D_{i,j}(-\partial_j(a)\partial_i(b)) + D_{j,i}(-\partial_i(a)\partial_j(b)) \\ &= D_{i,j}(\{a, b\}_{i,j}).\end{aligned}$$

(iii) For the action equation, we have

$$D_{i,j}(a)(b) = \big(\partial_i(a)\partial_j - \partial_j(a)\partial_i\big)(b) = \partial_i(a)\partial_j(b) - \partial_j(a)\partial_i(b) = \{a, b\}_{i,j},$$

as required. In particular, a is in the center of $A, \{\ ,\ \}_{i,j}$ if and only if $D_{i,j}(a)(A) = 0$ and hence if and only if $D_{i,j}(a) = 0$. Certainly, this occurs if and only if $a \in \ker \theta_{i,j}$ and consequently if and only if $\partial_i(a) = \partial_j(a) = 0$. □

Observe that $W = A\Delta$ has an obvious A-module structure. We use this in the following result which is somewhat stronger than we actually require. We will really only use the $v = 1$ case.

Lemma 2.3. *Let $\alpha \in W$, $v \in A$ and let i, j be distinct elements of the index set $\mathcal{I}$. If $[\alpha, D_{i,j}(vA)] = 0$, then $v\alpha = 0$.*

Proof. For convenience, let $i = 1$, $j = 2$ and write $\alpha = a_1\partial_1 + a_2\partial_2 + \cdots + a_n\partial_n$. The goal is to show that $va_k = 0$ for all k. Let $x \in A$ and note that $D_{1,2}(vx) = \partial_1(vx)\partial_2 - \partial_2(vx)\partial_1$. Thus, by considering the ∂_1-coefficient of $0 = [\alpha, D_{1,2}(vx)]$, we obtain

$$\partial_2(vx)\partial_1(a_1) - \partial_1(vx)\partial_2(a_1) - \sum_{k=1}^{n} a_k\partial_k\partial_2(vx) = 0 \qquad \text{for all } x \in A.$$

Furthermore, by expanding $\partial_1(vx)$, $\partial_2(vx)$ and $\partial_k\partial_2(vx)$ and by grouping terms, we get

$$\sum_{k=1}^{n} va_k\partial_k\partial_2(x) = \sum_{k=1}^{n} b_k\partial_k(x) + cx \qquad \text{for all } x \in A,$$

where $b_k, c \in A$ are elements not depending upon x. Setting $x = 1$ in the above shows that $c = 0$. Furthermore, replacing x by xy and expanding the derivatives yields

$$\begin{aligned}
\sum_{k=1}^{n} va_k(\partial_k(x)\partial_2(y) + \partial_2(x)\partial_k(y)) \\
&= x\cdot\Big(\sum_{k=1}^{n} b_k\partial_k(y) - \sum_{k=1}^{n} va_k\partial_k\partial_2(y)\Big) \\
&\quad + y\cdot\Big(\sum_{k=1}^{n} b_k\partial_k(x) - \sum_{k=1}^{n} va_k\partial_k\partial_2(x)\Big) \\
&= x\cdot 0 + y\cdot 0 = 0 \qquad \text{for all } x, y \in A.
\end{aligned}$$

Thus, for fixed $x \in A$, the element

$$\beta = \sum_{k=1}^{n} va_k(\partial_k(x)\partial_2 + \partial_2(x)\partial_k) \in W$$

acts trivially on A. But, we know that W acts faithfully on A, so $\beta = 0$ and hence $v_k a_k\partial_2(x) = 0$ for all $x \in A$ and $k \neq 2$. Again, the faithfullness of the action yields $va_k = 0$ for all $k \neq 2$. In a similar manner, by considering the ∂_2-coefficient of $0 = [\alpha, D_{1,2}(vx)]$, we conclude that $va_k = 0$ for all $k \neq 1$. $\square$

Now, let us assume for the remainder of this section that condition (2) of Theorem 1.6 holds. In other words, $\partial_i(A^j) \neq 0$ and $\partial_i(A^{j,k}) \neq 0$ for all i, j, k.

Lemma 2.4. *Let V be a K-subspace of A.*

i. *Let $\partial \in \Delta$, and suppose that there exists a nonzero Δ-stable subspace X of A with $X\partial(V) \subseteq V$. Then AV is a ∂-stable ideal of A. Furthermore, if $X \subseteq A^{i,j}$, then $A{\cdot}\{V,A\}_{i,j}$ and $A{\cdot}\{V,V\}_{i,j}$ are ∂-stable ideals of A.*
ii. *$\{V,A\}_{i,j} \supseteq \partial_j(A^i)\partial_i(V) = \partial_i\big(\partial_j(A^i)V\big)$. In particular, $\{A,A\}_{i,j} \supseteq \partial_i(A)$, and if V is a Lie ideal of $A, \{\ ,\ \}_{i,j}$, then AV, $A{\cdot}\{V,A\}_{i,j}$ and $A{\cdot}\{V,V\}_{i,j}$ are all $\{\partial_i,\partial_j\}$-stable associative ideals of A.*

Proof. (i) For the first ideal, it suffices to show that $A\partial(V) \subseteq AV$. To this end, note that $XA = A$ since A is Δ-simple. Thus $A\partial(V) = AX\partial(V) \subseteq AV$, as required. For the last two ideals, it suffices to show that $X\partial\big(\{V,A\}_{i,j}\big) \subseteq \{V,A\}_{i,j}$ and that $X\partial\big(\{V,V\}_{i,j}\big) \subseteq \{V,V\}_{i,j}$ if $X \subseteq A^{i,j}$. In the former case, Lemma 2.2(i) yields

$$\begin{aligned} X\partial\big(\{V,A\}_{i,j}\big) &\subseteq X\{\partial(V),A\}_{i,j} + X\{V,\partial(A)\}_{i,j} \\ &\subseteq \{X\partial(V),A\}_{i,j} + \{V,X\partial(A)\}_{i,j} \subseteq \{V,A\}_{i,j}, \end{aligned}$$

since $X \subseteq A^{i,j}$. The latter inclusion follows by replacing A by V in the above.

(ii) First, observe that $\{V,A\}_{i,j} \supseteq \{V,A^i\}_{i,j} = \partial_j(A^i)\partial_i(V) = \partial_i\big(\partial_j(A^i)V\big)$. In particular, $\{A,A\}_{i,j} \supseteq \partial_i\big(\partial_j(A^i)A\big) = \partial_i(A)$ since $\partial_j(A^i)A$ is a nonzero Δ-stable ideal of A. Furthermore, if V is a Lie ideal of $A, \{\ ,\ \}_{i,j}$, then $V \supseteq \{V,A\}_{i,j}$, so $V \supseteq \partial_j(A^i)\partial_i(V)$ and (i) implies that AV is ∂_i-stable. Similarly, it is ∂_j-stable. Since $\{V,A\}_{i,j}$ and $\{V,V\}_{i,j}$ are also Lie ideals of $A, \{\ ,\ \}_{i,j}$, the result follows. □

If $\partial_i\partial_i(A) = 0$, that is, if $\partial_i(A) \subseteq A^i$, then it is easy to see that A^i is a Δ-stable commutative Lie ideal of $A, \{\ ,\ \}_{i,j}$ which can certainly be larger than $A^{i,j}$. Moreover if $c \in A^\Delta$ and if $\partial = \partial_i + c\partial_j \in \mathrm{Der}_K(A)$, then the Poisson bracket $\{\ ,\ \}_{i,j}$ determined by ∂_i and ∂_j is equal to the bracket determined by ∂ and ∂_j. Thus if $\partial^2 = 0$, then A^∂ is also a Δ-stable commutative Lie ideal of $A, \{\ ,\ \}_{i,j}$. Aspects of the following argument, without assuming (ii), can be used to obtain a partial converse to this fact. However, we just pursue the proof far enough to obtain what we need. At this point, condition (3) of Theorem 1.6 comes into play. Thus we assume that either $\mathrm{char}\, K \neq 2$ or that $\partial_i(\Delta(A^j)) \neq 0$ for all distinct i,j.

Lemma 2.5. *Let V be a Lie ideal of $A, \{\ ,\ \}_{i,j}$ not contained in the center $A^{i,j}$ and assume that*

i. *For each $\partial \in \mathcal{B}$ there exists a nonzero Δ-stable subspace X of $A^{i,j}$ with $X\partial(V) \subseteq V$.*
ii. *$\partial_i(V)\Delta(A^i) \subseteq V$ and $\partial_j(V)\Delta(A^j) \subseteq V$ if $\mathrm{char}\, K = 2$.*

Then $V \supseteq \{A,A\}_{i,j}$.

Proof. By (i) and Lemma 2.4(i), we know that the ideals $I = \{V,V\}_{i,j}{\cdot}A$ and $J = \{V,A\}_{i,j}{\cdot}A$ are Δ-stable and hence equal to either 0 or A. Furthermore, since V is not contained in $A^{i,j}$, it follows that $J \neq 0$ and hence that $J = A$ is not nil.

But $\{V,A\}_{i,j} \subseteq \partial_i(V)A + \partial_j(V)A$, so we see that one of $\partial_i(V)$ or $\partial_j(V)$ does not consist of nilpotent elements. Say $\partial_j(A)$ is not nil.

The goal now is to show that $\{V,V\}_{i,j} \neq 0$ and, of course, this follows from Lemma 2.1 and $J = A$ if $\operatorname{char} K \neq 2$. Thus suppose that $\operatorname{char} K = 2$ and assume, by way of contradiction, that $\{V,V\}_{i,j} = 0$. Since $\partial_j(V)$ is not nil, we can choose $b \in V$ with $0 \neq \partial_j(b)^2 \in A^\Delta$, by Lemma 1.4(iv), and hence $\partial_j(b)$ is invertible in A. Now, for all $a \in V$, we have $\partial_i(a)\partial_j(b) - \partial_j(a)\partial_i(b) = \{a,b\}_{i,j} \in \{V,V\}_{i,j} = 0$, so $\partial_i(a) = c\partial_j(a)$ where $c = \partial_i(b)\partial_j(b)^{-1} \in A$ is independent of a. We first observe that $c \in A^\Delta$. To this end, let $\partial \in \mathcal{B}$ and let X be given by (i). If $a \in V$ and $x \in X \subseteq A^{i,j}$, then $x\partial(a) \in V$, so $x\partial\partial_i(a) = \partial_i(x\partial(a)) = c\partial_j(x\partial(a)) = cx\partial\partial_j(a)$. On the other hand, by applying ∂ to $\partial_i(a) = c\partial_j(a)$ and multiplying by x, we obtain $x\partial\partial_i(a) = x\partial(c)\partial_j(a) + cx\partial\partial_j(a)$. Thus $x\partial(c)\partial_j(a) = 0$, so $\partial(c)X\partial_j(V) = 0$ and hence $\partial(c) = 0$ since X is Δ-stable and $\partial_j(V)$ contains the unit $\partial_j(b)$. Finally, by (ii), $\partial_j(V)\Delta(A^j) \subseteq V$. Thus if $a \in V$ and $y \in \Delta(A^j) \subseteq A^j$, then $\partial_j(a)y \in V$ and

$$\partial_i(y)\partial_j(a) + y\partial_i\partial_j(a) = \partial_i(\partial_j(a)y) = c\partial_j(\partial_j(a)y) = cy\partial_j\partial_j(a).$$

On the other hand, by applying ∂_j to $\partial_i(a) = c\partial_j(a)$ and multiplying by y, we obtain $y\partial_i\partial_j(a) = cy\partial_j\partial_j(a)$ since $c \in A^\Delta \subseteq A^j$. Thus, we conclude that $\partial_i(y)\partial_j(a) = 0$, so $\partial_i(\Delta(A^j))\partial_j(V) = 0$. But $\partial_i(\Delta(A^j))$ is a nonzero Δ-stable subset of A by condition (3) of Theorem 1.6, so $\partial_j(V) = 0$, a contradiction.

We conclude, therefore, that $\{V,V\}_{i,j} \neq 0$, so $I \neq 0$ and consequently $I = A$. Lemma 2.1 now yields $V \supseteq \{A,I\}_{i,j} = \{A,A\}_{i,j}$, as required. $\square$

For convenience, we isolate below some consequences of hypothesis (4b), using the notation of Lemma 1.5.

Lemma 2.6. *Let $\partial_i \in \mathcal{B}$ be diagonalizable on A, write $\mathcal{T}_i = \mathcal{T}_{\partial_i}$, and suppose that either* $\operatorname{char} K = p > 0$ *or that $1 \in \Delta(A^i)^n$ for two relatively prime integers n. If L is a Lie ideal of $D = \mathbb{D}(A,\Delta)$, then $\mathcal{T}_i(L) \subseteq L$.*

Proof. Let $X = \Delta(A^i)$ and observe by (2) that X is a nonzero Δ-stable subspace of A^i. Furthermore, since $X\partial_i \subseteq D$ by Lemma 1.3(i) and since L is a Lie ideal of D, it follows that $(\operatorname{ad}_{X\partial_i})^n(L) \subseteq L$ for all $n \geq 1$. Note that ∂_i is diagonalizable in its action on A and hence $\mathcal{T}_i$ is diagonalizable in its action on $W = A\Delta$.

Suppose first that $\operatorname{char} K = p > 0$. Since $XA = A$, it follows that X cannot consist of nilpotent elements. In particular, Lemma 1.4(iv) implies that $X^p \cap A^\Delta \neq 0$. But A^Δ is a field and X is certainly an A^Δ-subspace of A, so $X^p \supseteq A^\Delta$ and consequently $1 \in X^p$. By Lemma 1.5, $\mathcal{T}_i^p \in (\operatorname{ad}_{X\partial_i})^p$ and hence $\mathcal{T}_i^p(L) \subseteq L$. Since $\mathcal{T}_i$ is diagonalizable and since the pth power map on K is one-to-one, it follows that $\mathcal{T}_i$ and $\mathcal{T}_i^p$ have the same eigenspaces on $A\Delta$. In particular, any $\mathcal{T}_i^p$-stable subspace of $A\Delta$ is $\mathcal{T}_i$-stable, and we conclude that $\mathcal{T}_i(L) \subseteq L$.

On the other hand, suppose that $1 \in \Delta(A^i)^r$ and $1 \in \Delta(A^i)^s$ with r and s relatively prime integers ≥ 1. Then, by Lemma 1.5, $\mathcal{T}_i^r \in (\operatorname{ad}_{X\partial_i})^r$ and $\mathcal{T}_i^s \in$

$(\mathrm{ad}_{X\partial_i})^s$, so $\mathcal{T}_i^r(L) \subseteq L$ and $\mathcal{T}_i^s(L) \subseteq L$. Now r and s are relatively prime, so there exist positive integers u and v with $1 = ru - sv$, and clearly $\mathcal{T}_i^{ru}(L) \subseteq L$ and $\mathcal{T}_i^{sv}(L) \subseteq L$. Furthermore, since $\mathcal{T}_i$ is diagonalizable, it is both invertible and locally algebraic in its action on W/W_0 where $W_0 = \ker \mathcal{T}_i$. Thus $\mathcal{T}_i^{sv}(L) + W_0 \subseteq L + W_0$ implies that $\mathcal{T}_i^{sv}(L) + W_0 = L + W_0$, so

$$\mathcal{T}_i(L) = \mathcal{T}_i(L + W_0) = \mathcal{T}_i\big(\mathcal{T}_i^{sv}(L) + W_0\big) = \mathcal{T}_i^{1+sv}(L) = \mathcal{T}_i^{ru}(L) \subseteq L,$$

and the lemma is proved. $\square$

The following key result essentially reduces the question of simplicity to the 2-dimensional case. We will need to assume condition (4) of Theorem 1.6 at this point in the argument. Recall, from Lemma 1.3(i), that $D \supseteq \Delta(A^i)\partial_i$ for all i and note that $\Delta(A^i) \supseteq \partial_j(A^i) \neq 0$ by (2).

Lemma 2.7. *Let L be a nonzero Lie ideal of D. Then $L \cap (A\partial_i + A\partial_j) \neq 0$ for all distinct i, j.*

Proof. Since $L \neq 0$, we can choose $n \geq 2$ minimal so that, by relabeling elements of the basis $\mathcal{B}$, there exist $\partial_1, \partial_2, \ldots, \partial_n \in \mathcal{B}$ with $i = 1, j = 2$ and with

$$L_n = L \cap (A\partial_1 + A\partial_2 + \cdots + A\partial_n) \neq 0.$$

We suppose that $n \geq 3$ (in particular, $\dim_K \Delta \geq 3$) and derive a contradiction. Let V denote the set of ∂_n-coefficients of elements of L_n. Then V is a nonzero K-subspace of A. If $V \cap A^{1,k} \neq 0$ for some $k = 2, 3, \ldots, n-1$, choose an element $\alpha \in L_n$ with ∂_n-coefficient a nonzero member of $A^{1,k}$. Then it is easy to see that $[D_{1,k}(A), \alpha] \subseteq L_{n-1}$ and hence we must have $[D_{1,k}(A), \alpha] = 0$ by the minimality of n. But $\alpha \neq 0$, so this contradicts Lemma 2.3. In other words, $V \cap A^{1,k} = 0$ for all such k. Furthermore, since $[D_{1,2}(A), L_n] \subseteq L_n$, we have $\{A, V\}_{1,2} = D_{1,2}(A)(V) \subseteq V$ and hence V is a Lie ideal of $A, \{\ ,\ \}_{1,2}$. We proceed in a series of two steps.

Step 1. $n = 3$ *and* $V \cap A^1 \neq 0$.

Proof. We first show that V satisfies the hypotheses of Lemma 2.5 with $i = 1$ and $j = 2$. To start with, we know that V is a Lie ideal of $A, \{\ ,\ \}_{1,2}$ which is not contained in $A^{1,2}$. Furthermore, since $\Delta(A^1)\partial_1 \subseteq D$, we have $[\Delta(A^1)\partial_1, L_n] \subseteq L_n$ and hence $\Delta(A^1)\partial_1(V) \subseteq V$. Similarly, $\Delta(A^2)\partial_2(V) \subseteq V$, so (ii) of Lemma 2.5 holds. To prove (i), there are two cases to consider according to which condition in (4) we assume is satisfied.

Let $\ell \in \mathcal{I}$ be arbitrary, and suppose first that (4a) holds. If $\mathcal{I}' = \{1, 2, \ldots, n\}$, then (4a) implies that $X = \Delta(A^\ell) \cap A^{\mathcal{I}'}$ is a nonzero Δ-stable subspace of $A^{1,2}$, and of course $X\partial_\ell \subseteq D$. Thus $[X\partial_\ell, L_n] \subseteq L$ and in fact $[X\partial_\ell, L_n] \subseteq L_n$ since $X \subseteq A^{\mathcal{I}'}$. It follows that $X\partial_\ell(V) \subseteq V$, as required. On the other hand, if (4b) holds, then by Lemma 2.6 and its notation, we have $\mathcal{T}_\ell(L) \subseteq L$. Hence, $\mathcal{T}_\ell(L_n) \subseteq L_n$, and therefore $K\partial_\ell(V) = \partial_\ell(V) \subseteq V$.

We can now conclude from Lemma 2.5 that $V \supseteq \{A, A\}_{1,2}$ and hence that $V \supseteq \partial_2(A)$ by Lemma 2.4(ii). In particular, $V \supseteq \partial_2(A^{1,3})$, and $V \cap A^{1,3} \neq 0$ by (2). The observation of the first paragraph now implies that 3 cannot be smaller than n, so $n = 3$ and this step is proved. $\square$

Step 2. *Final contradiction.*

Proof. By Step 1, there exists $\alpha = a\partial_1 + b\partial_2 + c\partial_3 \in L_3 = L_n$ with c a nonzero element of A^1. Note that $\partial_2(A^1)\partial_1 \subseteq \Delta(A^1)\partial_1 \subseteq D$, so $[\alpha, \partial_2(A^1)\partial_1] \subseteq L_3$. But the ∂_3-coefficients here are 0 since $\partial_1(c) = 0$ and hence the minimality of $n = 3$ implies that $[\alpha, \partial_2(A^1)\partial_1] = 0$. In particular, if $x \in A^1$, then by considering the ∂_1-coefficient of $0 = [\alpha, \partial_2(x)\partial_1]$, we obtain

$$b\partial_2\partial_2(x) + c\partial_3\partial_2(x) - \partial_1(a)\partial_2(x) = 0 \qquad \text{for all } x \in A^1$$

since $\partial_2(x) \in A^1$. If $y \in A^{1,2}$, then $xy \in A^1$, so the above displayed equation holds when x is replaced by xy. Since y is also a constant for ∂_2, this new displayed equation quickly simplifies to

$$c\partial_3(y)\partial_2(x) = -y\cdot(b\partial_2\partial_2(x) + c\partial_3\partial_2(x) - \partial_1(a)\partial_2(x)) = y\cdot 0 = 0.$$

In other words, $c\partial_3(A^{1,2})\partial_2(A^1) = 0$. But $\partial_3(A^{1,2})$ and $\partial_2(A^1)$ are nonzero Δ-stable subsets of A, so this forces c to equal 0, a contradiction. $\square$

We can now offer the

Proof of Theorem 1.6. Since $D \supseteq [S, S] \neq 0$, it clearly suffices to show that D is Lie simple. To this end, let L be a nonzero Lie ideal of D and let i, j be distinct subscripts. By Lemma 2.7, we know that $L' = L \cap (A\partial_i + A\partial_j) \neq 0$, and hence by Lemma 2.3, $[D_{i,j}(A), L'] \neq 0$. Since $[D_{i,j}(A), L'] \subseteq D_{i,j}(A)$, by Lemma 1.2(iii), it follows that $L \cap D_{i,j}(A) \neq 0$, and certainly the latter intersection is a Lie ideal of $D_{i,j}(A)$. Now let $L_{i,j}$ be defined by $L_{i,j} = \{a \in A \mid D_{i,j}(a) \in L\}$. We show that $L_{i,j}$ satisfies the hypotheses of Lemma 2.5. First, since $L_{i,j}$ is the complete inverse image under $\theta_{i,j}$ of $L \cap D_{i,j}(A) \neq 0$, it follows from Lemma 2.2(ii) that $L_{i,j}$ is a Lie ideal of $A, \{\ ,\ \}_{i,j}$ properly containing $A^{i,j}$. Next, since $D_{i,j}(L_{i,j}) \subseteq L$ and $\Delta(A^i)\partial_i \subseteq D$, we see that $L \supseteq [\Delta(A^i)\partial_i, D_{i,j}(L_{i,j})]$ and, by Lemma 1.2(iii), the latter expression is equal to $D_{i,j}\big(\Delta(A^i)\partial_i(L_{i,j})\big)$. Thus $\Delta(A^i)\partial_i(L_{i,j}) \subseteq L_{i,j}$ and similarly $\Delta(A^j)\partial_j(L_{i,j}) \subseteq L_{i,j}$ and (ii) is proved. For (i), we again consider the assumptions (4a) and (4b) separately.

Let $\ell \in \mathcal{I}$ be arbitrary, possibly equal to i or j. If (4a) is satisfied then $X = \Delta(A^\ell) \cap A^{i,j} \neq 0$. Since $X\partial_\ell \subseteq D$, it follows that $[x\partial_\ell, D_{i,j}(a)] \in L$ for all $a \in L_{i,j}$ and $x \in X$. In particular, since $x \in A^{i,j}$, this implies that

$$\begin{aligned} D_{i,j}(x\partial_\ell(a)) &= \partial_i(x\partial_\ell(a))\partial_j - \partial_j(x\partial_\ell(a))\partial_i \\ &= x\partial_\ell\partial_i(a)\partial_j - x\partial_\ell\partial_j(a)\partial_i = [x\partial_\ell, D_{i,j}(a)] \end{aligned}$$

is contained in $L \cap D_{i,j}(A)$ and hence that $x\partial_\ell(a) \in L_{i,j}$. Thus, $X\partial_\ell(L_{i,j}) \subseteq L_{i,j}$, as required. On the other hand, if (4b) holds, then by Lemma 2.6 and its notation, we have $\mathcal{T}_\ell(L) \subseteq L$. Hence, $D_{i,j}(L_{i,j}) \supseteq \mathcal{T}_\ell\big(D_{i,j}(L_{i,j})\big) = D_{i,j}\big(\partial_\ell(L_{i,j})\big)$, so $L_{i,j} \supseteq \partial_\ell(L_{i,j}) = K\partial_\ell(L_{i,j})$. Thus (i) is proved and Lemma 2.5 implies that $L_{i,j} \supseteq \{A, A\}_{i,j}$. In particular, $D_{i,j}(A)/\big(L \cap D_{i,j}(A)\big) \cong A/L_{i,j}$ is abelian.

Now Lemma 2.4(ii) implies that $\{A, A\}_{i,j} \supseteq \partial_i(A)$ and $\partial_j(A)$, so we have $L_{i,j} \supseteq \partial_i(A) + \partial_j(A) + A^{i,j}$. Thus, if (5a) holds, then $L \cap D_{i,j}(A) = D_{i,j}(L_{i,j}) = D_{i,j}(A)$, so $L \supseteq D_{i,j}(A)$ for all i, j and consequently $L = D$, as required. On the other hand, suppose (5b) is satisfied. Since $L_{i,j} \supseteq \partial_j(A)$, we see that $L \supseteq D_{i,j}\big(\partial_j(A) \cap A^j\big) = \partial_i\big(\partial_j(A) \cap A^j\big)\partial_j$. Thus, $1\partial_j \in L$ for all $j \in \mathcal{I}$, and hence $1\Delta \subseteq L$. Next, observe that $L \supseteq [1\Delta, D_{i,j}(A)] = D_{i,j}\big(\Delta(A)\big)$. In particular, if we have $A = \Delta(A) + A^{i,j}$, then $L \supseteq D_{i,j}\big(\Delta(A)\big) = D_{i,j}\big(\Delta(A) + A^{i,j}\big) = D_{i,j}(A)$ for all i, j, and $L = D$.

Finally, suppose $\dim_K \Delta \geq 3$ and let i, j, k be any three distinct elements of $\mathcal{I}$. Then $1 \in \partial_i\big(\partial_j(A) \cap A^j\big)$ implies that $1 \in \partial_i(A^j)$, and hence $L_{i,j} \supseteq \partial_i(A) \supseteq \partial_i(A^jA^i) \supseteq \partial_i(A^j)A^i \supseteq A^i$. In particular, if $X = \{a \in A^i \mid \partial_j(a) \in A^j\}$, then $X \subseteq A^i \subseteq L_{i,j}$ and $L \supseteq D_{i,j}(X) = \partial_j(X)\partial_i$. Note that $Y = \partial_j(X) = \partial_j(A^i) \cap A^j \subseteq A^j$ and that $L \supseteq [Y\partial_i, D_{j,k}(A)]$. Now if $y \in Y$ and $a \in A$, then Lemma 1.2(iii) yields

$$[y\partial_i, D_{j,k}(a)] = [y\partial_i, d_j(a)\partial_k - \partial_k(a)\partial_j] = D_{i,k}\big(y\partial_j(a)\big) - D_{i,j}\big(y\partial_k(a)\big).$$

Of course, $[y\partial_i, D_{j,k}(a)] \in L$ and also $D_{i,k}\big(y\partial_j(a)\big) = D_{i,k}\big(\partial_j(ya)\big) \subseteq D_{i,k}\big(\Delta(A)\big) \subseteq L$ since $y \in A^j$. Thus $D_{i,j}\big(y\partial_k(a)\big) \in L$ for all $y \in Y$ and $a \in A$. But $Y\partial_k(A) = A$ by (5b), so we conclude that $D_{i,j}(A) \subseteq L$ for all i, j, and therefore $L = D$. □

§3. Examples

In this final section, we consider a number of applications of the main theorem. We begin with several well-known examples which have the additional property that $1 \in \Delta(A^i)$ for all $i \in \mathcal{I}$. Specifically, let us assume that $1 \in \partial_i\big(\partial_j(A) \cap A^j\big)$. Then $1 \in \partial_i(A^j) \subseteq \Delta(A^j)$ and hence, since $\Delta(A^j) \cap A^\Delta \neq 0$, condition (4a) is automatically satisfied. Thus, Theorem 1.6 simplifies to

Corollary 3.1. *Let K be a field, let A be a commutative K-algebra, and let Δ be a K-subspace of $\mathrm{Der}_K(A)$ of dimension ≥ 2 consisting of commuting derivations. Write $S = \mathbb{S}(A, \Delta)$, $D = \mathbb{D}(A, \Delta)$, and let $\mathcal{B} = \{\partial_i \mid i \in \mathcal{I}\}$ be a fixed K-basis for Δ. Assume that*

- (1′) *A is Δ-simple and $A^\Delta\Delta$ acts faithfully on A.*
- (2′) *$1 \in \partial_i\big(\partial_j(A) \cap A^j\big)$ for all i, j, and $\partial_i(A^{j,k}) \neq 0$ for all distinct i, j, k.*
- (3′) *Either $\mathrm{char}\, K \neq 2$ or $\partial_i(\Delta(A^j)) \neq 0$ for all distinct i, j.*
- (4′) *One of the following two conditions is satisfied.*
    - (a) *$A = \Delta(A) + A^{i,j}$ for all distinct i, j.*
    - (b) *$\dim_K \Delta \geq 3$ and $\big(\partial_i(A^j) \cap A^i\big)\cdot\partial_k(A) = A$ for all i, j, k.*

Then $D = [S, S]$ is Lie simple.

We remark that, if $1 \in \Delta(A^i)$ for all i, then $(1')$ is actually a necessary condition. Indeed, suppose D is Lie simple. Since $\Delta(A^i)\partial_i \subseteq D$ and $\Delta(A^i)$ is an A^Δ-submodule of A, we see that $1\Delta \subseteq D$ and also that $A^\Delta\Delta \subseteq D$. From the first inclusion, we deduce that D must act faithfully on A, since Δ does, and from the second we conclude that $A^\Delta\Delta$ acts faithfully on A. Next, let I be a nonzero Δ-stable associative ideal of A. Then $\Delta(I) \neq 0$ since otherwise $1 \in \Delta(A^1)$ implies that $0 = \Delta(I) \supseteq \Delta(IA^1) = I\Delta(A^1) \supseteq I$. In particular, if $\partial_i(I) \neq 0$, then $0 \neq D_{i,j}(I) \subseteq I\Delta \cap D$, and therefore $I\Delta \cap D$ is a nonzero Lie ideal of D. But D is Lie simple, so $I\Delta \cap D = D$ and hence $I\Delta \supseteq D \supseteq \Delta(A^1)\partial_1 \supseteq 1\partial_1$, so $1 \in I$ and A is Δ-simple.

We can use this corollary to handle the rather standard examples in characteristic 0 of polynomial rings $K[x_i \mid i \in \mathcal{I}]$, group rings $K[x_i^{\pm} \mid i \in \mathcal{I}]$, and formal power series rings $K[[x_i \mid i \in \mathcal{I}]]$. Furthermore, in characteristic $p > 0$, we consider truncated polynomial rings $K[x_i \mid x_i^p = 0,\, i \in \mathcal{I}]$, group rings $K[x_i \mid x_i^p = 1,\, i \in \mathcal{I}]$, and twisted group rings $K[x_i \mid x_i^p = k_i,\, i \in \mathcal{I}]$ where $0 \neq k_i \in K$. In all of these cases, the derivations are the usual partial derivatives with respect to the variables. We could also easily consider mixed variable cases, like $K[x_1, x_2, \ldots][y_1^{\pm}, y_2^{\pm}, \ldots]$ or even $K[x_1, x_2, \ldots][[y_1, y_2, \ldots]]$. However, we have chosen to avoid introducing these additional complications into the statement below.

Example 3.2. *Let K be a field and let $\{x_i \mid i \in \mathcal{I}\}$ be a family of variables. Define $\partial_i = \partial/\partial x_i$ and set $\Delta = \sum_{i\in\mathcal{I}} K\partial_i$. Assume that one of the following three situations occurs.*

i. $\operatorname{char} K = 0$, $A = K[x_i \mid i \in \mathcal{I}]$ *or* $K[[x_i \mid i \in \mathcal{I}]]$, *and* $|\mathcal{I}| \geq 2$.
ii. $\operatorname{char} K = 0$, $A = K[x_i^{\pm} \mid i \in \mathcal{I}]$, *and* $|\mathcal{I}| \geq 3$.
iii. $\operatorname{char} K = p > 0$, $A = K[x_i \mid x_i^p \in K,\, i \in \mathcal{I}]$, *and* $|\mathcal{I}| \geq 3$.

Then $D = \mathbb{D}(A, \Delta)$ is Lie simple.

Proof. We verify that the hypotheses of Corollary 3.1 are satisfied. To start with, it is clear that Δ acts faithfully on A and that $A^\Delta = K$. Thus $A^\Delta\Delta = \Delta$ acts faithfully on A. Next, let I be a nonzero Δ-stable ideal of A. If $A \neq K[[x_i \mid i \in \mathcal{I}]]$ as in case (i), then we can assume that I contains a nonzero polynomial in the variables $\{x_i \mid i \in \mathcal{I}\}$. By repeated differentiation, it is clear that I contains a nonzero constant polynomial and hence $I = A$. On the other hand, if $A = K[[x_i \mid i \in \mathcal{I}]]$, then by repeated differentiation we see that I contains an element a with a nonzero constant term. But this implies that a is a unit in A, and hence $I = A$ again. Thus $(1')$ holds in all cases.

For condition $(2')$ we see that $x_i = \partial_j(x_i x_j) \in \partial_j(A) \cap A^j$ and hence that $1 = \partial_i(x_i) \in \partial_i\big(\partial_j(A) \cap A^j\big)$. Furthermore, $1 = \partial_i(x_i) \in \partial_i(A^{j,k})$. Next, for $(3')$, we need only assume that $\operatorname{char} K = 2$ and hence that $|\mathcal{I}| \geq 3$. Here we see that $x_i = \partial_k(x_i x_k) \in \Delta(A^j)$, so $1 = \partial_i(x_i) \in \partial_i\big(\Delta(A^j)\big)$, as required.

It remains to consider ($4'$). To start with, if $A = K[x_i \mid i \in \mathcal{I}]$ or $K[[x_i \mid i \in \mathcal{I}]]$ as in (i), then $\partial_i(A) = A$, so ($4'$a) is satisfied. Next, suppose that A is given by (ii) or (iii), so that $\dim_K \Delta \geq 3$. Here, let $i, j, k \in \mathcal{I}$, take $\sigma \in A^{i,j}$ and let $\tau \in A^k$. Then $\partial_k(A)$ contains $\partial_k(\tau x_k) = \tau$, and $\partial_i(A^j) \cap A^i$ contains $\partial_i(x_i\sigma) = \sigma \in A^i$. Thus $\big(\partial_i(A^j) \cap A^i\big)\cdot\partial_k(A)$ contains $\sigma\tau$, and hence $\big(\partial_i(A^j) \cap A^i\big)\cdot\partial_k(A) \supseteq A^{i,j}A^k = A$, since $A^{i,j}$ contains the K-linear span of all powers of x_k and since A^k is the K-span of all monomials in variables other than x_k. Corollary 3.1 now yields the result. □

It is clear that the above proof is quite different from the graded Lie algebra argument found, for example, in [SF, Chapter 3]. As we will see later on, the Lie algebras D corresponding to cases (ii) or (iii), but with $|\mathcal{I}| = 2$, are definitely not simple. Indeed, they satisfy $D \neq [D, D]$.

Now we turn to the special algebra analog of the Witt-type example given in [KW]. This is also related to examples found in [O] and [DZ]. Basically, it involves a diagonal action of Δ on either the group algebra $K[G]$ of an abelian group G or on a commutative twisted group algebra $K^t[G]$. It is perhaps best to formulate this result in terms of G-graded rings.

Let G be a multiplicative group and recall that a K-algebra A is G-graded if $A = \oplus \sum_{x\in G} A_x$ is a direct sum of the K-subspaces A_x, indexed by the elements $x \in G$, and with $A_xA_y \subseteq A_{xy}$ for all $x, y \in G$. It is easy to see that $1 \in A_1$ and that A_1 is a subalgebra of A. Furthermore, A is said to be strongly G-graded if $A_xA_y = A_{xy}$ for all $x, y \in G$. As is well known, this occurs if and only if $A_xA_{x^{-1}} = A_1$ for all $x \in G$. Note that any twisted group algebra $K^t[G]$ is strongly G-graded with components given by $K^t[G]_x = K\bar{x}$. It is easy to see that in any strongly graded ring, the left annihilator of each component A_x is equal to 0. Indeed, if $aA_x = 0$, then $a \in aA_1 = aA_xA_{x^{-1}} = 0$.

Next, we say that $\lambda\colon G \to K$ is a K-functional on G if λ is a homomorphism from G to the additive group K^+ of K. Thus λ is a K-functional if and only if $\lambda(xy) = \lambda(x) + \lambda(y)$ for all $x, y \in G$. It is clear that $\mathrm{Hom}(G, K^+)$, the set of all such λ, is a K-vector space, having the obvious addition and scalar multiplication. Furthermore, if $G^\lambda = \ker_G \lambda = \{x \in G \mid \lambda(x) = 0\}$, then certainly $G^\lambda \triangleleft G$ and G/G^λ is isomorphic to a subgroup of K^+. In particular, G/G^λ is torsion-free abelian if $\mathrm{char}\, K = 0$, and it is an elementary abelian p-group if $\mathrm{char}\, K = p > 0$.

Finally, we note that each $\lambda \in \mathrm{Hom}(G, K^+)$ gives rise to a derivation $\lambda^\#$ on A by defining

$$\lambda^\#\colon \sum_{x\in G} a_x \mapsto \sum_{x\in G} \lambda(x)a_x$$

where $a_x \in A_x$. Indeed, the map ${}^\#\colon \mathrm{Hom}(G, K^+) \to \mathrm{Der}_K(A)$ is easily seen to be a vector space monomorphism provided that each component A_x of A is nonzero. In particular, this holds if A is strongly graded. Furthermore, because of the diagonal action, it is clear that $\mathrm{Hom}(G, K^+)^\#$, the image of $\mathrm{Hom}(G, K^+)$ consists of commuting derivations.

The following is a first approximation to our main result on the simplicity of special algebras determined by structures of this sort. Note that $1 \notin \Delta(A^i)$ in these examples.

Lemma 3.3. *Let G be a multiplicative group and let $A = \oplus \sum_{x\in G} A_x$ be a strongly G-graded commutative K-algebra with $A_1 = K$. Furthermore, let Λ be a K-subspace of $\operatorname{Hom}(G, K^+)$ with $\dim_K \Lambda \geq 2$ and set $\Delta = \Lambda^\# \subseteq \operatorname{Der}_K(A)$. Now suppose $\mathcal{L} = \{\lambda_i \mid i \in \mathcal{I}\}$ is a K-basis for Λ and write $G^i = G^{\lambda_i} = \ker_G \lambda_i$ for each $i \in \mathcal{I}$. If $\bigcap_{\ell\in\mathcal{I}} G^\ell = 1$, $G^i \neq G^i \cap G^j$ and $G^i \cap G^j \neq G^i \cap G^j \cap G^k$ for all distinct i, j, k, then $D = \mathbb{D}(A, \Delta)$ is Lie simple.*

Proof. For convenience, let us write $G^{i,j} = G^i \cap G^j$ and $G^{i,j,k} = G^i \cap G^j \cap G^k$. Note that all components A_x of A are nonzero since A is strongly graded. In particular, if we set $\partial_i = \lambda_i^\#$ for each $i \in \mathcal{I}$, then $\mathcal{B} = \{\partial_i \mid i \in \mathcal{I}\}$ is a basis for Δ. Now if X is a subset of G, let $A(X)$ be defined by $A(X) = \oplus \sum_{x\in X} A_x$. Then $A = A(G)$, and it is easy to see that $A^i = A(G^i)$, $A^{i,j} = A(G^{i,j})$ and $\partial_i(A) = A(G \setminus G^i)$. Furthermore, since $\bigcap_{i\in\mathcal{I}} G^i = 1$, it is clear that $\Delta(A^i) = A(G^i \setminus 1)$ and that $A^\Delta = A_1 = K$.

We consider the hypotheses of Theorem 1.6. To start with, we have $\partial_i(A^j) = \partial_i\big(A(G^j)\big) = A(G^j \setminus G^{i,j}) \neq 0$ since $G^j \supsetneq G^{i,j}$. Next, we note that $\partial_i(A^{j,k}) = \partial_i\big(A(G^{j,k})\big) = A(G^{j,k} \setminus G^{i,j,k}) \neq 0$ since $G^{j,k} \supsetneq G^{i,j,k}$, and therefore (2) is satisfied. In addition, we have $\partial_i\big(\Delta(A^j)\big) = \partial_i\big(A(G^j \setminus 1)\big) = A(G^j \setminus G^{i,j}) \neq 0$, so (3) holds. Finally, note that

$$\partial_i(A) + \partial_j(A) + A^{i,j} = A(G \setminus G^i) + A(G \setminus G^j) + A(G^{i,j}) = A(G) = A$$

and hence (5a) is satisfied.

If $a = \sum_{x\in G} a_x \in A$, then we recall that $\operatorname{supp} a = \{x \in G \mid a_x \neq 0\}$. Thus, the support of a is a finite subset of G which is nonempty precisely when $a \neq 0$. Now suppose that I is a nonzero associative ideal of A and let a be an element of minimal support size in $I \setminus 0$. Say $S = \operatorname{supp} a$ and $a = \sum_{x\in S} a_x$. If $y^{-1} \in S$, then $A_y a \subseteq I$ and each element in $A_y a$ has support contained in yS. Furthermore, since A is strongly graded, there exists $0 \neq b \in A_y a \subseteq I$, and the minimality of $|S|$ implies that $\operatorname{supp} b = yS$. Finally, note that $1 \in yS$, since $y^{-1} \in S$, and that $|yS| = |S|$. Thus we conclude that for any such ideal I there always exists a nonzero element of minimal support size having 1 in its support.

It remains to consider (1) and (4b). For (1), we have $A^\Delta \Delta = K\Delta = \Delta$ and, as we have observed, $\Delta = \Lambda^\#$ acts faithfully on A. In addition, if I is a nonzero Δ-stable ideal of A, choose a to be a nonzero element of I having minimal support size and having 1 in its support. Now, for any $\partial \in \Delta$, we have $\partial(a) \in I$ and clearly $\operatorname{supp} \partial(a) \subseteq \operatorname{supp} a \setminus 1$. Thus, the minimality of $|\operatorname{supp} a|$ implies that $\partial(a) = 0$ for all such ∂. In other words, $0 \neq a \in A^\Delta = K$, so $a \in I$ is invertible and $I = A$. Consequently, A is Δ-simple and (1) is satisfied. Finally, the diagonal action implies that (4b) holds when $\operatorname{char} K = p > 0$. On the other hand, if $\operatorname{char} K = 0$, then

$\bigcap_{\ell \in \mathcal{I}} G^\ell = 1$ implies that G is torsion free. Furthermore, $G^i \neq 1$, so this subgroup has infinite order. It then follows easily from the strong grading and $|G_i| > 2$ that $\Delta(A^i)^2 = A(G^i \setminus 1)^2 = A(G^i)$ and therefore that $1 \in A(G^i) = A(G^i \setminus 1)^n = \Delta(A^i)^n$ for all integers $n \geq 2$. Thus (4b) holds in all characteristics, and Theorem 1.6 yields the result. $\square$

To proceed further, we need to choose an appropriate basis for Λ.

Lemma 3.4. *Let G be a multiplicative group and let Λ be a subspace of the K-vector space* $\mathrm{Hom}(G, K^+)$.

i. *If $\dim_K \Lambda < \infty$, then there exists a basis $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for Λ and elements $x_1, x_2, \ldots, x_n \in G$ such that $\lambda_i(x_j) = \delta_{i,j}$, the Kronecker delta.*
ii. *Suppose $\bigcap_{\lambda \in \Lambda} \ker_G \lambda = 1$, let H be a finitely generated subgroup of G and let Ω be a finite dimensional K-subspace of Λ. Then there exists a finitely generated subgroup $\bar{H}$ of G with $\bar{H} \supseteq H$ and a finite dimensional K-subspace $\bar{\Omega}$ of Λ with $\bar{\Omega} \supseteq \Omega$ such that the restriction $\bar{\Omega}_{\bar{H}}$ of $\bar{\Omega}$ to $\bar{H}$ yields an embedding into* $\mathrm{Hom}(\bar{H}, K^+)$ *with $\bigcap_{\lambda \in \bar{\Omega}} \ker_{\bar{H}} \lambda = 1$.*

Proof. (i) We proceed by induction on $\dim_K \Lambda$. Assume that $\dim_K \Lambda \geq 1$ and let μ be a nonzero member of this space. Then there exists $x_1 \in G$ with $\mu(x_1) \neq 0$ and, by replacing μ by $\mu(x_1)^{-1}\mu$ if necessary, we can assume that $\mu(x_1) = 1$. Clearly $\Lambda = K\mu \oplus \Lambda'$ where $\Lambda' = \{\lambda \in \Lambda \mid \lambda(x_1) = 0\}$. By induction, there exists a basis $\{\lambda_2, \ldots, \lambda_n\}$ for Λ' and there are elements $x_2, \ldots, x_n \in G$ with $\lambda_i(x_j) = \delta_{i,j}$. Of course, we also have $\lambda_i(x_1) = 0$. Finally, if we replace μ by $\lambda_1 = \mu - \sum_2^n \mu(x_i)\lambda_i$, then $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a basis for Λ with $\lambda_i(x_j) = \delta_{i,j}$ for all $i = 1, 2, \ldots, n$.

(ii) Since Ω is finite dimensional, it follows from (i) above that there is a basis $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of Ω and elements $x_1, x_2, \ldots, x_n \in G$ such that $\lambda_i(x_j) = \delta_{i,j}$. Now let $\bar{H} = \langle H, x_1, x_2, \ldots, x_n \rangle$ be the subgroup of G generated by H and the elements $x_1, x_2, \ldots, x_n$. Then $\bar{H}$ is finitely generated and, since $x_1, x_2, \ldots, x_n \in \bar{H}$, it follows that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are K-linearly independent in their action on $\bar{H}$. In other words, $\Omega \cong \Omega_{\bar{H}}$. Now, by assumption, $\bigcap_{\lambda \in \Lambda_{\bar{H}}} \ker_{\bar{H}} \lambda = 1$. Furthermore, if $\operatorname{char} K = p > 0$, then $\bar{H}$ is a finite elementary abelian p-group, and if $\operatorname{char} K = 0$, then $\bar{H}$ is a free abelian group of finite rank and hence it satisfies the descending chain condition on pure subgroups. It follows in either case that the images $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n \in \Omega_{\bar{H}}$ can be extended to a finite K-linearly independent set $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n, \ldots, \tilde{\lambda}_t\} \subseteq \Lambda_{\bar{H}}$ such that $\bigcap_{i=1}^t \ker_{\bar{H}} \tilde{\lambda}_i = 1$. In particular, if $\lambda_{n+1}, \ldots, \lambda_t \in \Lambda$ are inverse images of $\tilde{\lambda}_{n+1}, \ldots, \tilde{\lambda}_t \in \Lambda_{\bar{H}}$, then it is clear that $\{\lambda_1, \ldots, \lambda_n, \ldots, \lambda_t\}$ is a K-linearly independent subset of Λ. Consequently, if we let $\bar{\Omega}$ be the K-subspace of Λ spanned by these t elements, then $\bar{\Omega}$ is a finite dimensional subspace containing Ω. Moreover, it acts faithfully on $\bar{H}$ and $\bigcap_{\lambda \in \bar{\Omega}} \ker_{\bar{H}} \lambda = 1$. $\square$

With this, we can prove

Theorem 3.5. *Let G be a multiplicative group and let $A = \oplus\sum_{x\in G} A_x$ be a strongly G-graded commutative K-algebra with $A_1 = K$. Furthermore, let Λ be a K-subspace of* $\mathrm{Hom}(G, K^+)$ *with* $\dim_K \Lambda \geq 2$ *and set* $\Delta = \Lambda^\# \subseteq \mathrm{Der}_K(A)$. *If* $\bigcap_{\lambda\in\Lambda} \ker_G \lambda = 1$*, then* $D = \mathbb{D}(A, \Delta)$ *is locally simple with respect to* G *and* Λ*, and hence it is Lie simple.*

Proof. Suppose first that Λ is finite dimensional. Then, by Lemma 3.4(i), there is a basis $\mathcal{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of Λ and elements $x_1, x_2, \ldots, x_n \in G$ such that $\lambda_i(x_j) = \delta_{i,j}$. In particular, using the notation of Lemma 3.3, we have $\bigcap_1^n G^i = \bigcap_{\lambda\in\Lambda} \ker_G \lambda = 1$. Furthermore, if i, j and k are distinct subscripts, then $x_j \in G^i$ but $x_j \notin G^i \cap G^j$, and $x_k \in G^i \cap G^j$ but $x_k \notin G^i \cap G^j \cap G^k$. Thus $G^i \neq G^i \cap G^j$ and $G^i \cap G^j \neq G^i \cap G^j \cap G^k$, so Lemma 3.3 implies that D is Lie simple.

For the general case, if H is a subgroup of G, recall that $A(H) = \sum_{x\in H} A_x$ is an H-graded subalgebra of $A(G) = A$. Furthermore, if Ω is a subspace of Λ, recall that Ω_H is the restriction of $\Omega \subseteq \mathrm{Hom}(G, K^+)$ to H, so that $\Omega_H \subseteq \mathrm{Hom}(H, K^+)$. In particular, if the natural map $\Omega \to \Omega_H$ is a bijection, then it is clear that $\mathbb{W}(A(H), \Omega_H^\#) \subseteq \mathbb{W}(A, \Lambda^\#)$ and hence that $\mathbb{D}(A(H), \Omega_H^\#) \subseteq \mathbb{D}(A, \Lambda^\#)$.

Now let S be any finite collection of elements of $\mathbb{D}(A, \Lambda^\#)$. Then there exists a finitely generated subgroup H of G and finitely many linearly independent elements $\mu_1, \mu_2, \ldots, \mu_n \in \Lambda$ with S contained in $\sum_{i,j} D_{\mu_i,\mu_j}(A(H))$. In particular, if Ω is the K-linear span of $\mu_1, \mu_2, \ldots, \mu_n$, then $\dim_K \Omega = n < \infty$ and Lemma 3.4(ii) applies. It follows that there exists a finitely generated subgroup $\bar{H}$ of G containing H and a finite dimensional K-subspace $\bar{\Omega}$ of Λ containing Ω such that $\bar{\Omega} \cong \bar{\Omega}_{\bar{H}}$ and $\bigcap_{\lambda\in\bar{\Omega}} \ker_{\bar{H}} \lambda = 1$. As we observed previously, the bijection $\bar{\Omega} \cong \bar{\Omega}_{\bar{H}}$ implies that $S \subseteq \mathbb{D}(A(\bar{H}), \bar{\Omega}_{\bar{H}}^\#)$. Furthermore, since $\bigcap_{\lambda\in\bar{\Omega}} \ker_{\bar{H}} \lambda = 1$, the conclusion of the first paragraph implies that $\mathbb{D}(A(\bar{H}), \bar{\Omega}_{\bar{H}}^\#)$ is Lie simple. In other words, D is locally simple with respect to G and Λ, and this certainly implies that it is Lie simple. □

Note that in the above, if $a \in A_x$ and $\lambda, \mu \in \Lambda$, then

$$D_{\lambda,\mu}(a) = \lambda^\#(a)\mu^\# - \mu^\#(a)\lambda^\# = \lambda(x)a\mu^\# - \mu(x)a\lambda^\# \in A_x \otimes \Delta.$$

In particular, since $\lambda(1) = 0$ for all $\lambda \in \Lambda$, it follows that $D \cap (A_1 \otimes \Delta) = 0$. But $A_1 \otimes \Delta \subseteq S = \mathbb{S}(A, \Delta)$, so we conclude that S properly contains D and therefore that S is not Lie simple.

Example 3.6. *Let K be a field and let $\{x_i \mid i \in \mathcal{I}\}$ be a family of variables with $|\mathcal{I}| \geq 2$. Define $\partial_i = x_i{\cdot}(\partial/\partial x_i)$ and set $\Delta = \sum_{i\in\mathcal{I}} K\partial_i$. Assume that either*

i. $\mathrm{char}\, K = 0$ *and* $A = K[x_i^\pm \mid i \in \mathcal{I}]$, *or*
ii. $\mathrm{char}\, K = p > 0$ *and* $A = K[x_i \mid x_i^p \in K \setminus 0,\ i \in \mathcal{I}]$.

Then $D = \mathbb{D}(A, \Delta)$ is Lie simple.

Proof. In case (i), we note that $A = K[G]$ where G is the free abelian group with generators $\{x_i \mid i \in \mathcal{I}\}$. On the other hand, in case (ii), we see that $A =$

$K^t[G]$ is a twisted group algebra where G is the elementary abelian p-group with "free" generators $\{x_i \mid i \in \mathcal{I}\}$. Furthermore, in the notation of the preceding corollary, $\partial_i = \lambda_i^{\#}$ where $\lambda_i: G \to K^+$ is the K-functional given by $\lambda_i(x_i) = 1$ and $\lambda_i(x_j) = 0$ for all $j \neq i$. Since the set $\{\lambda_i \mid i \in \mathcal{I}\}$ is dual to the "basis" $\{x_i \mid i \in \mathcal{I}\}$ for G, it is clear that $\{\lambda_i \mid i \in \mathcal{I}\}$ is a K-linearly independent subset of $\mathrm{Hom}(G, K^+)$. Furthermore, if Λ is the linear span of $\{\lambda_i \mid i \in \mathcal{I}\}$, then $\dim_K \Lambda \geq 2$ and $\bigcap_{\lambda \in \Lambda} \ker_G \lambda = 1$. We conclude from Theorem 3.5 that D is Lie simple. □

We can, of course, construct more exotic examples of this sort. Indeed, for each $i \in \mathcal{I}$, let R_i be a nonzero additive subgroup of K^+ and define G, in additive notation, to be a subdirect product of the R_i's. Then, for each $i \in \mathcal{I}$, there exists an epimorphism $\lambda_i: G \to R_i \subseteq K$ and we conclude from Theorem 3.5 that $\mathbb{D}(A, \Delta)$ is Lie simple. Here, of course, $A = K[G]$, $\Lambda = \sum_{i \in \mathcal{I}} K\lambda_i$ has dimension at least 2, and $\Delta = \Lambda^{\#}$.

Next, we return to the general context and take a closer look at the dimension 2 situation. The following is a sharpened version of Theorem 1.6 in this special case.

Proposition 3.7. *Let K be a field, let A be a commutative K-algebra, and let Δ be a K-subspace of $\mathrm{Der}_K(A)$ of dimension 2 consisting of commuting derivations. Write $S = \mathbb{S}(A, \Delta)$, $D = \mathbb{D}(A, \Delta)$, and let $\mathcal{B} = \{\partial_1, \partial_2\}$ be a fixed K-basis for Δ. Assume that*

- (1″) *A is Δ-simple and $A^{\Delta}\Delta$ acts faithfully on A.*
- (2″) *$\partial_i(A^j) \neq 0$ for all $\{i, j\} = \{1, 2\}$.*
- (3″) *Either $\mathrm{char}\, K \neq 2$ or $\partial_i(\Delta(A^j)) \neq 0$ and $\Delta(A^i) \cap A^j \neq 0$ for all i, j.*
- (4″) *$\partial_1(A) + \partial_2(A) + A^{1,2} = A$.*

Then $D = [S, S]$ is Lie simple.

Proof. Let L be a nonzero Lie ideal of $D = D_{1,2}(A)$. Then, by Lemma 2.2, $L = D_{1,2}(V)$ where $V = \{a \in A \mid D_{1,2}(a) \in L\}$ is a Lie ideal of $A, \{\ ,\ \}_{1,2}$ properly containing $A^{1,2}$. The goal is to show that $V \supseteq \{A, A\}_{1,2}$. Suppose first that $\mathrm{char}\, K \neq 2$. Then Lemma 2.4(ii) implies that $I = \{V, V\}_{1,2} \cdot A$ and $J = \{V, A\}_{1,2} \cdot A$ are Δ-stable ideals of A. Furthermore, $J \neq 0$ since V is properly larger than $A^{1,2}$, and therefore $J = A$. Lemma 2.1 now implies that $I \neq 0$, so $I = A$ and we conclude that $V \supseteq \{A, I\}_{1,2} = \{A, A\}_{1,2}$.

On the other hand, if $\mathrm{char}\, K = 2$, then by (3″) we can assume that $\partial_i\big(\Delta(A^j)\big) \neq 0$ and that $\Delta(A^i) \cap A^j \neq 0$ for all i, j. Here we show that V satisfies the hypotheses of Lemma 2.5. To start with, by Lemma 2.4(ii), we have $V \supseteq \{V, A\}_{1,2} \supseteq \partial_2(A^1)\partial_1(V) = \Delta(A^1)\partial_1(V)$, and similarly $V \supseteq \Delta(A^2)\partial_2(V)$, This proves (ii), and (i) follows since $\Delta(A^i) \supseteq \Delta(A^i) \cap A^j \neq 0$. Thus, by Lemma 2.5, we can again conclude that $V \supseteq \{A, A\}_{1,2}$. Finally, by Lemma 2.4(ii) and (4″), we have $V \supseteq \{A, A\}_{1,2} \supseteq \partial_1(A) + \partial_2(A) + A^{1,2} = A$, so $L = D_{1,2}(V) = D_{1,2}(A) = D$. □

Finally, we briefly comment on the relationship between Examples 3.2 and 3.6. Let A, Δ and $\mathcal{B} = \{\partial_i \mid i \in \mathcal{I}\}$ be given, and suppose that, for each i, u_i is a unit of

A which is a constant for all ∂_j with $j \neq i$. Then $\mathcal{B}' = \{u_i\partial_i \mid i \in \mathcal{I}\}$ is easily seen to be a set of commuting derivations of A, and we let Δ' denote its K-linear span. If $\mathcal{B}'$ is linearly independent (necessarily true if $A^\Delta\Delta$ acts faithfully on A), then we can form the two Witt type algebras $W = A \otimes \Delta$ and $W' = A \otimes \Delta'$, expecting to obtain two quite different structures. Instead, we wind up with only one structure, since these two Lie algebras are always isomorphic. Indeed, if W acts faithfully on A, then W and W' are the same subset of $\mathrm{Der}_K(A)$ and, in any case, the map $W' \to W$ given by $a \otimes u_i\partial_i \mapsto au_i \otimes \partial_i$ is easily seen to be a Lie isomorphism. On the other hand, this specific isomorphism does not preserve the divergence map and hence the relationship between the corresponding special algebras is less clear. In fact, $\mathbb{D}(A, \Delta)$ and $\mathbb{D}(A, \Delta')$ need not be isomorphic in general.

For example, suppose that either $\mathrm{char}\, K = 0$ and $A = K[x^\pm, y^\pm]$ or $\mathrm{char}\, K = p > 0$ and $A = K[x, y \mid x^p, y^p \in K \setminus 0]$. Then x and y are units of A and we can consider the two sets of commuting derivations given by $\mathcal{B} = \{\partial/\partial x, \partial/\partial y\}$ and $\mathcal{B}' = \{x(\partial/\partial x), y(\partial/\partial y)\}$. Thus $D = \mathbb{D}(A, \Delta)$ is the algebra of Example 3.2(ii)(iii), but with $|\mathcal{I}| = 2$, and $D' = \mathbb{D}'(A, \Delta)$ is the algebra given by Example 3.6(i)(ii), again with $|\mathcal{I}| = 2$. Furthermore, the Lie algebras here with the same base ring A are not isomorphic. Indeed, by Example 3.6, we know that D' is Lie simple, and therefore this fact will follow from

Example 3.8. *Let K be a field, let $\{x, y\}$ be variables and let Δ be the K-linear span of $\partial/\partial x$ and $\partial/\partial y$. If either* $\mathrm{char}\, K = 0$ *and* $A = K[x^\pm, y^\pm]$, *or* $\mathrm{char}\, K = p > 0$ *and $A = K[x, y \mid x^p, y^p \in K]$, then $D = \mathbb{D}(A, \Delta)$ is not Lie simple. Indeed, we have $D \supsetneq [D, D] \supsetneq 0$.*

Proof. Let $\{\,,\,\}$ denote the Poisson bracket determined by the Jacobian associated with $\partial/\partial x$ and $\partial/\partial y$. Then we know that there is a Lie epimorphism from $A, \{\,,\,\}$ to D with kernel $A^\Delta = K$. We compute $\{A, A\}$. To this end, let $x^m y^n$ and $x^r y^s$ be monomials in A. Then

$$\begin{aligned}\{x^m y^n, x^r y^s\} &= \frac{\partial}{\partial x}(x^m y^n)\cdot\frac{\partial}{\partial y}(x^r y^s) - \frac{\partial}{\partial y}(x^m y^n)\cdot\frac{\partial}{\partial x}(x^r y^s)\\ &= \begin{vmatrix} m & n \\ r & s \end{vmatrix} \cdot x^{m+r-1} y^{n+s-1}\\ &= \begin{vmatrix} a+1 & b+1 \\ r & s \end{vmatrix} \cdot x^a y^b,\end{aligned}$$

where $a = m + r - 1$ and $b = n + s - 1$. By taking $r = 1,\ s = 0$ or $r = 0,\ s = 1$, we see that $\{A, A\}$ contains all monomials $x^a y^b$ with both $a \neq -1$ (or $a \neq p - 1$ if $\mathrm{char}\, K = p > 0$) and $b \neq -1$ (or $b \neq p - 1$ if $\mathrm{char}\, K = p > 0$). Furthermore, from the form of the determinant, it is clear that the remaining monomial $x^{-1}y^{-1}$ (or $x^{p-1}y^{p-1}$ if $\mathrm{char}\, K = p > 0$) does not occur in $\{A, A\}$. Thus, $A \supsetneq \{A, A\} \supsetneq K$, and we conclude that $D \supsetneq [D, D] \supsetneq 0$, as required. $\square$

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Department of Mathematics, DePaul University, Chicago, Illinois 60614
E-mail address: jbergen@condor.depaul.edu

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706
E-mail address: passman@math.wisc.edu