JOURNAL OF ALGEBRA 176, 417–448 (1995)

Derivations of Skew Polynomial Rings

J. Marshall Osborn* and D. S. Passman*

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

Communicated by Susan Montgomery

Received February 22, 1994

Let F be a field of characteristic 0 and let $\lambda_{i,j} \in F^{\cdot}$ for $1 \le i, j \le n$. Define $R = F[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n]$ to be the skew polynomial ring with $\bar{x}_i\bar{x}_j = \lambda_{i,j}\bar{x}_j\bar{x}_i$ and let $S = F[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n, \bar{x}_1^{-1}, \bar{x}_2^{-1}, \dots, \bar{x}_n^{-1}]$ be the corresponding Laurent polynomial ring. In a recent paper, Kirkman, Procesi, and Small considered these two rings under the assumption that S is simple and showed, for example, that the Lie ring of inner derivations of S is simple. Furthermore, when $n = 2$, they determined the automorphisms of S, related its ring of inner derivations to a certain Block algebra, and proved that every derivation of R is the sum of an inner derivation and a derivation which sends each x_i to a scalar multiple of itself. In this paper, we extended these results to a more general situation. Specifically, we study twisted group algebras $F^t[G]$ where G is a commutative group and F is a field of any characteristic. Furthermore, we consider certain subalgebras $F^t[H]$ where H is a subsemigroup of G which generates G as a group. Finally, if $e: G \times G \to F$ is a skew-symmetric bilinear form, then we study the Lie algebra $F_e[G]$ associated with e, and we consider its relationship to the Lie structure defined on various twisted group algebras $F^t[G]$.

1. TWISTED GROUP ALGEBRAS

Let G be an arbitrary multiplicative group and let F be a field. Then a twisted group ring $F^t[G]$ is an associative F-algebra with F-basis $\overline{G}$, a copy of G, and with multiplication defined distributively using

$$\bar{x}\bar{y} = t(x, y)\overline{xy} \qquad \text{for all } x, y \in G.$$

Here t: $G \times G \to F^{\cdot} = F \setminus \{0\}$ is the twisting function and associativity is easily seen to be equivalent to

$$t(x, y)t(xy, z) = t(x, yz)t(y, z) \qquad \text{for all } x, y, z \in G.$$

In other words, t must be a 2-cocycle. On the other hand, a simple (diagonal) change of basis, replacing each $\bar{x}$ by $\tilde{x} = d(x)\bar{x}$ with d: $G \to F^{\cdot}$,

* E-mail addresses: osborn@math.wisc.edu; passman@math.wisc.edu.

0021-8693/95 $12.00

obviously maintains the same structure but replaces the twisting t by

$$\bar{t}(x, y) = d(x)d(y)d(xy)^{-1}t(x, y).$$

Thus t and $\bar{t}$ are equivalent modulo a 2-coboundary and therefore the various twisted group algebras $F^t[G]$ are in one-to-one correspondence with the elements of the cohomology group $H^2(G, F^{\cdot})$. By way of a diagonal change of basis, we can assume, without loss of generality, that $\bar{1} = 1$ is the identity element of $F^t[G]$. Furthermore, $\mathscr{G} = \{a\bar{g} \mid a \in F^{\cdot}, g \in G\}$ is a group of units of $F^t[G]$, the so-called group of trivial units, and $\mathscr{G}/F^{\cdot} \cong G$. If $t(x, y) = 1$ for all $x, y \in G$, then $F^t[G]$ is untwisted and $F^t[G] = F[G]$ is the ordinary group algebra of G over F.

Again let $F^t[G]$ be given. If $\alpha \in F^t[G]$, then α is a finite sum $\alpha = \sum_g a_g \bar{g}$ with $a_g \in F$ and $g \in G$. We then let supp α, the support of α, be the finite subset of G consisting of those group elements g which occur in this sum with coefficient $a_g \neq 0$. Furthermore, if X is a subset of G, we let

$$F^t[X] = \{\alpha \in F^t[G] \mid \operatorname{supp} \alpha \subseteq X\}.$$

In particular, if H is a subsemigroup of G (with 1), then $F^t[H]$ is a subalgebra of $F^t[G]$. Now if x is any element of G, then we set $\mathbb{C}^t_G(x) = \{g \in G \mid \bar{g}\bar{x} = \bar{x}\bar{g}\}$ so that $\mathbb{C}^t_G(x)$ is clearly a subgroup of G contained in $\mathbb{C}_G(x)$. Similarly, we set

$$\mathbb{Z}^t(G) = \{g \in G \mid \bar{g}\bar{x} = \bar{x}\bar{g} \quad \text{for all } x \in G\}$$

so that $\mathbb{Z}^t(G)$ is a subgroup of $\mathbb{Z}(G)$.

At this point and throughout the remainder of the paper we assume that G is an abelian group. Thus

$$\bar{x}\bar{y} = \lambda(x, y)\bar{y}\bar{x} \qquad \text{for all } x, y \in G,$$

where $\lambda\colon G \times G \to F^{\cdot}$ is a map which is easily seen to satisfy

$$\lambda(x, yz) = \lambda(x, y)\lambda(x, z) \qquad \text{and} \qquad \lambda(y, x) = \lambda(x, y)^{-1}$$

for all $x, y, z \in G$. Notice that

$$[\bar{x}, \bar{y}] = \bar{x}\bar{y} - \bar{y}\bar{x} = (1 - \lambda(y, x))\bar{x}\bar{y} = (1 - \lambda(y, x))t(x, y)\overline{xy}$$

for all $x, y \in G$. In particular, $\operatorname{supp}[\bar{x}, \bar{y}] = \{xy\}$ if $\bar{x}$ and $\bar{y}$ do not commute and the support is empty otherwise. Since $xy \in \mathbb{Z}^t(G)$ implies that $\bar{x}$ and $\bar{y}$ commute, it follows immediately that $\operatorname{supp}[\bar{x}, \bar{y}]$ is always disjoint from $\mathbb{Z}^t(G)$.

Now let $G^\# = G \setminus 1$ be the set of nonidentity elements of G. Then the preceding remark implies that $\mathscr{L}(G) = F^t[G^\#]$ is a Lie subalgebra of $F^t[G]$. Furthermore, if $\mathscr{L}(G)$ is Lie simple, then clearly either $\mathbb{Z}^t(G) = 1$, or $|G| = 2$ and $\dim_F \mathscr{L}(G) = 1$.

LEMMA 1.1. *Let $F^t[G]$ be given.*

(i) *$\mathbb{Z}F^t[G]$, the center of $F^t[G]$, is equal to $F^t[Z]$ where $Z = \mathbb{Z}^t(G)$. In particular, $\mathbb{Z}F^t[G] = F$ if and only if $\mathbb{Z}^t(G) = 1$.*

(ii) *If Z is free abelian or if F is algebraically closed, then $F^t[Z] \cong F[Z]$.*

Proof. (i) It is clear that $F^t[Z] \subseteq \mathbb{Z}F^t[G]$. For the converse, let $\alpha = \sum_g a_g \bar{g} \in \mathbb{Z}F^t[G]$. Then, for any $x \in G$,

$$0 = [\alpha, \bar{x}] = \sum_g a_g(1 - \lambda(x, g))\bar{g}\bar{x}$$

so $\lambda(x, g) = 1$ for all $g \in \operatorname{supp} \alpha$. Thus $\operatorname{supp} \alpha \subseteq Z$, and $\alpha \in F^t[Z]$.

(ii) Let $\mathscr{Z} = \{a\bar{z} \mid a \in F^\cdot, z \in Z\}$ be the group of trivial units of the commutative algebra $F^t[Z]$. Then $\mathscr{Z}$ is a commutative group with $\mathscr{Z}/F^\cdot \cong Z$. If F is algebraically closed, then $F^\cdot$ is divisible and hence injective. If Z is free abelian, then Z is projective. In either case, $\mathscr{Z}/F^\cdot \cong Z$ splits, so $\mathscr{Z} = F^\cdot \times X$ where X is a complementary subgroup isomorphic to Z. Since the elements of X form an untwisted group basis for $F^t[Z]$, we conclude that $F^t[Z] = F^t[X] = F[X] \cong F[Z]$. ∎

For any ring R, let InDer R denote its Lie ring of inner derivations. We can now prove

LEMMA 1.2. *Let $F^t[G]$ be given and assume that either*

(i) *$F^t[G]$ is a simple ring, or*

(ii) *InDer $F^t[G]$ is a nonzero simple Lie algebra.*

Then $\mathbb{Z}F^t[G]$ is a field. In particular, if F is algebraically closed or if G is free abelian, then $\mathbb{Z}F^t[G] = F$ and $\mathbb{Z}^t(G) = 1$.

Proof. If $F^t[G]$ is a simple ring, then certainly $\mathbb{Z}F^t[G]$ is a field.

On the other hand, suppose InDer $F^t[G]$ is a nonzero simple Lie algebra. Then the nonzero hypothesis implies that $F^t[G]$ is noncommutative and therefore G properly contains $\mathbb{Z}^t(G) = Z$. Now let α be a nonzero element of $\mathbb{Z}F^t[G] = F^t[Z]$ and observe that $\alpha F^t[G]$ is a two-sided ideal of $F^t[G]$ which therefore determines a Lie ideal of InDer $F^t[G]$. Since $G \neq Z$, it is clear that $\alpha F^t[G]$ cannot be contained in the center of $F^t[G]$ and therefore $\alpha F^t[G]$ must determine all of InDer $F^t[G]$. As a consequence, $\alpha F^t[G] + F^t[Z] = F^t[G]$. Now choose a fixed element $x \in G \setminus Z$ and apply the natural projection $F^t[G] \to F^t[Zx]$ to the previous equality. Then $\alpha F^t[Zx] = F^t[Zx] = F^t[Z]\bar{x}$, so it follows that $\alpha F^t[Z] = F^t[Z]$. Thus α is invertible in $F^t[Z]$, and $F^t[Z]$ is a field in this case also.

Finally, if G is free abelian, then so is Z. Thus if either G is free abelian or F is algebraically closed, then Lemma 1.1(ii) implies that $F^t[Z] \cong F[Z]$. But $F[Z]$ has an augmentation homomorphism onto F determined by sending the group basis to 1. In particular, if $F^t[Z]$ is a field, then the kernel of this homomorphism must be trivial and therefore $Z = 1$, as required. ∎

Observe that if F is the field of rational numbers and if γ is a complex nth root of 2, then $F[\gamma]$ is a field which is easily seen to be isomorphic to a twisted group algebra over F of the cyclic group G of order n. In particular, $\mathbb{Z}F^t[G]$ can be a field without having $\mathbb{Z}^t(G) = 1$. Of course, in this case, G is not free abelian and F is not algebraically closed.

For the remainder of this section we assume that $\mathbb{Z}^t(G) = 1$ and therefore that $\mathbb{Z}F^t[G] = F$. Let H be a subsemigroup of G (with 1) and suppose that H generates G as a group. Then certainly $F^t[H]$ generates $F^t[G]$ and therefore $\mathbb{Z}F^t[H]$ is also equal to F. Now let $H^\# = H \setminus 1$ be the set of nonidentity elements of H and, as before, write $\mathscr{L}(H) = F^t[H^\#]$. Then $\mathscr{L}[H]$ is a Lie subalgebra of $F^t[H]$, and $F^t[H] = \mathbb{Z}F^t[H] \oplus \mathscr{L}(H)$, so it is clear that InDer $F^t[H]$ is Lie isomorphic to $\mathscr{L}(H)$.

PROPOSITION 1.3. *Let $F^t[G]$ be a twisted group ring with $\mathbb{Z}^t(G) = 1$ and let H be a subsemigroup of G which generates G as a group.*

(i) *Any nonzero Lie ideal of $\mathscr{L}(H)$ contains $\bar{h}$ for some $h \in H^\#$.*

(ii) *Any nonzero ideal of $F^t[H]$ contains $\bar{h}$ for some element $h \in H$. In particular, $F^t[H]$ is a prime ring.*

Proof. (i) Let I be a nonzero Lie ideal of $L = \mathscr{L}(H)$ and let m be the minimal support size of the nonzero elements of I. Our goal is to show that $m = 1$. To this end, let $\alpha = \sum_{i=1}^{m} a_i \bar{x}_i$ be any element of I of support size m. If $g \in \mathbb{C}^t_H(x_1)$, then $[\alpha, \bar{g}] \in I$ has support size less than m and therefore this Lie commutator must be 0. Thus $\lambda(g, x_i) = 1$ for all i and all such g. In particular, since $x_1 \in \mathbb{C}^t_H(x_1)$, it follows that $\bar{x}_1$ commutes with all $\bar{x}_i$. In other words, the elements of $\overline{\operatorname{supp} \alpha}$ all commute in $F^t[H]$. Now suppose that $g \in H/\mathbb{C}^t_H(x_1)$. Then $[\alpha, \bar{g}] \in I$ and

$$[\alpha, \bar{g}] = \sum_{i=1}^{m} a_i(1 - \lambda(g, x_i))\bar{x}_i\bar{g}$$

is nonzero, so this Lie commutator must also have support size m. Therefore, by the above argument, the supporting elements of $[\alpha, \bar{g}]$ commute in $F^t[H]$. In other words, $\bar{x}_i\bar{g}\bar{x}_j\bar{g} = \bar{x}_j\bar{g}\bar{x}_i\bar{g}$ for all i, j and, since $\bar{g}\bar{x}_i = \lambda(g, x_i)\bar{x}_i\bar{g}$ and $\bar{x}_i\bar{x}_j = \bar{x}_j\bar{x}_i$, it follows that $\lambda(g, x_i) = \lambda(g, x_j)$. We have therefore shown that $\lambda(g, x_i) = \lambda(g, x_j)$ for all i, j and all $g \in H$.

Viewing these elements in G, we see that $\lambda(g, x_i x_j^{-1}) = 1$ for all $g \in H$. Thus $\bar{x}_i \bar{x}_j^{-1}$ centralizes $F^t[H]$ and therefore it centralizes all of $F^t[G]$. In other words, $x_i x_j^{-1} \in \mathbb{Z}^t(G) = 1$, so $x_i = x_j$ for all i, j, and therefore m is indeed equal to 1.

(ii) Now let J be a nonzero ideal of $F^t[H]$, and assume that $1 \notin J$. Since J is clearly not central in $F^t[H]$, we see that $[J, F^t[H]]$ is a nonzero Lie ideal of $\mathscr{L}(H)$. Thus, by (i) above, $[J, F^t[H]] \subseteq J$ contains $\bar{h}$ for some $h \in H^{\#}$, and the result follows since each $\bar{h}$ is a regular element of the ring. ∎

As a consequence, we obtain the following generalization of [KPS, Theorem 3.1] and [MPe, Proposition 1.3]. Obviously, it is a converse to Lemma 1.2.

THEOREM 1.4. *Let $F^t[G]$ be a twisted group algebra with G abelian. If $\mathbb{Z}^t(G) = 1$, then $F^t[G]$ is a simple ring and* InDer $F^t[G] \cong \mathscr{L}(G)$ *is a simple Lie algebra.*

Proof. By Proposition 1.3(ii), any nonzero ideal of $F^t[G]$ contains a unit, and therefore $F^t[G]$ is a simple ring.

Now let I be a nonzero Lie ideal of $\mathscr{L}(G) \cong$ InDer $F^t[G]$. Then, by part (i) of the preceding proposition, I contains $\bar{z}$ for some $z \in G^{\#}$. Furthermore, if $y \in G \setminus \mathbb{C}_G^t(z)$, then the formula for $[\bar{y}, \bar{z}]$ implies that I contains $\overline{yz}$. But yz is a typical element of $X = G \setminus \mathbb{C}_G^t(z)$, so I contains $F^t[X]$. Finally, observe that X generates G as a group. In particular, if g is any element of $G^{\#}$, then $\bar{g}$ cannot centralize all of $\bar{X}$. Thus there exists $x \in X$ with $g \notin \mathbb{C}_G^t(x)$. The above argument now implies that $\bar{g} \in I$ and, with this, we conclude that $I = \mathscr{L}(G)$, and $\mathscr{L}(G)$ is Lie simple. ∎

A second application of Proposition 1.3 concerns the Martindale ring of quotients of $F^t[H]$. More precisely, if R is any prime ring, then there are three distinct Martindale rings of quotients which can be defined to extend R. These are the right, left, and symmetric versions, and basic properties of each can be found in [Pa, Sect. 10].

PROPOSITION 1.5. *Let $\mathbb{Z}^t(G) = 1$ and let H be a subsemigroup of G which generates G as a group. Then $F^t[G]$ is the right, left, and symmetric Martingale ring of quotients of $F^t[H]$. In particular, any automorphism or derivation of $F^t[H]$ extends uniquely to an automorphism or derivation of $F^t[G]$.*

Proof. The proofs are virtually identical for all three quotient rings, so we will consider only the symmetric version. To this end, let Q be the symmetric Martindale ring of quotients of the prime ring $F^t[H]$. If $x \in H$, then $xH = Hx$ implies that $\bar{x}$ is a nonzero normal element of $F^t[H]$ and therefore $\bar{x}$ is invertible in Q. Thus $Q \supseteq F^t[H]\bar{H}^{-1}$. Furthermore, since $\bar{H}$ consists of normal elements, the ring extension $F^t[H]\bar{H}^{-1}$ is uniquely determined by $F^t[H]$. Hence, since H generates G, it follows that $F^t[H]\bar{H}^{-1} = F^t[G]$, and therefore $Q \supseteq F^t[G]$. For the reverse inclusion, let q be any element of Q. Then, by definition of Q, there exists a nonzero two-sided ideal I of $F^t[H]$ with $qI \subseteq F^t[H]$. But, by Proposition 1.3(ii), I contains $\bar{y}$ for some $y \in H$, so $q\bar{y} \in F^t[H]$ and $q \in F^t[H]\bar{H}^{-1} = F^t[G]$. We conclude that $Q = F^t[G]$ and [Pa, Lemma 10.9] yields the result. ∎

2. PSEUDO-INNER DERIVATIONS

Again, let G be an abelian group and let $F^t[G]$ be an arbitrary twisted group algebra. For convenience, write $Z = \mathbb{Z}^t(G)$ so that $\mathbb{Z}F^t[G] = F^t[Z]$. As will be apparent, there are two types of F-derivations of $F^t[G]$ which are of particular interest; these are the central and the pseudo-inner derivations which we define below.

Let $\theta: G \to F^t[Z]^+$ be a homomorphism from the multiplicative group G to the additive group $F^t[Z]^+$. In other words, θ is a map satisfying $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in G$. If we define the F-linear operator $\partial = \partial_\theta$ by $\partial(\bar{x}) = \theta(x)\bar{x}$ for all $x \in G$, then it is easy to see that ∂ is an F-derivation of $F^t[G]$. Indeed, for any $x, y \in G$, we have

$$\begin{aligned} \partial(\bar{x}\bar{y}) &= \partial(t(x,y)\overline{xy}) = t(x,y)\theta(xy)\overline{xy} \\ &= (\theta(x) + \theta(y))\bar{x}\bar{y} = \partial(\bar{x})\bar{y} + \bar{x}\partial(\bar{y}) \end{aligned}$$

since $\theta(y)$ is central in the ring. Furthermore, $\partial(\bar{x}) \in F^t[Zx]$ for all $x \in G$. We call any such ∂ obtained in this way a central derivation. Of course, if $Z = 1$, then $\theta: G \to F^+$ and, in this case, we also call ∂ a scalar derivation.

Now let $F^t[[G]]$ denote the set of all possibly infinite formal sums $\gamma = \sum_g c_g\bar{g}$ with $g \in G$ and $c_g \in F$. Then $F^t[[G]]$ is no longer a ring, but with the obvious definition of multiplication determined by the twisting function t, it is at least an $F^t[G]$-bimodule. In particular, if $x \in G$, then

$$\begin{aligned} \operatorname{ad}_\gamma \bar{x} &= [\gamma, \bar{x}] = \gamma\bar{x} - \bar{x}\gamma \\ &= \sum_g c_g(1 - \lambda(x,g))\bar{g}\bar{x} \end{aligned}$$

is a well-defined element of $F^t[[G]]$. Furthermore, the Jacobi identity applied to each summand of γ shows that

$$\operatorname{ad}_\gamma \bar{x}\bar{y} = (\operatorname{ad}_\gamma \bar{x})\bar{y} + \bar{x}(\operatorname{ad}_\gamma \bar{y})$$

for all $x, y \in G$. In particular, if γ has the additional property that $\mathrm{ad}_\gamma\, \bar{x} \in F^t[G]$ for all $x \in G$, then $\mathrm{ad}_\gamma\colon F^t[G] \to F^t[G]$ determines an F-derivation of the algebra. We call any such derivation of $F^t[G]$ obtained in this manner a pseudo-inner derivation. Since the elements of $Z \cap \operatorname{supp}\gamma$ have no effect on the map ad_γ, we may always suppose that $Z \cap \operatorname{supp}\gamma = \varnothing$. Clearly, $\mathrm{ad}_\gamma\, \bar{x} \in F^t[(G \setminus Z)x]$ for all $x \in G$.

LEMMA 2.1. *Let $\gamma \in F^t[\![G]\!]$.*

(i) *ad_γ is a pseudo-inner derivation if and only if*

$$\left(G \setminus \mathbb{C}_G^t(x)\right) \cap \operatorname{supp}\gamma$$

is finite for all $x \in G$

If G has finitely many elements $x_1, x_2, \ldots, x_n$ with $\bigcap_1^n \mathbb{C}_G^t(x_i) = \mathbb{Z}^t(G)$, then any pseudo-inner derivation of $F^t[G]$ is inner. In particular, this applies when G is a finitely generated group.

Proof. Part (i) is immediate from the preceding formula for $\mathrm{ad}_\gamma\, \bar{x}$ and part (ii) follows from

$$G = Z \cup \bigcup_1^n \left(G \setminus \mathbb{C}_G^t(x_i)\right)$$

and the fact that $\operatorname{supp}\gamma$ meets each of these summands in a finite set. ∎

In general, $F^t[G]$ can admit pseudo-inner derivations which are not inner. For example, let G be free abelian on the infinitely many generators $x_1, x_2, \ldots, y_1, y_2, \ldots$ and assume that $F^\cdot$ contains an element λ of infinite multiplicative order. Define $F^t[G]$ to be the Laurent polynomial ring in the variables $\bar{x}_i$, $\bar{y}_j$ subject to the relations

$$\bar{x}_i \bar{y}_i = \lambda \bar{y}_i \bar{x}_i \qquad \text{for all } i = 1, 2, \ldots$$

and with all other generators commuting. Then it is easy to see that $\mathbb{Z}^t(G) = 1$, so $F^t[G]$ is simple. Furthermore, notice that if ∂ is any inner derivation of $F^t[G]$, then the elements $\bar{y}_i$ are eventually constant for ∂. Thus the element $\gamma = \sum_{i=1}^\infty \bar{x}_i \in F^t[\![G]\!]$ determines a pseudo-inner derivation ad_γ which is not inner.

THEOREM 2.2. *Let $F^t[G]$ be a twisted group algebra of the abelian group G over the field F. Then any F-derivation of $F^t[G]$ is uniquely the sum of a central and a pseudo-inner derivation.*

Proof. Let ∂ be an F-derivation of $F^t[G]$ and, for each $x \in G$, write $\partial(\bar{x})\bar{x}^{-1} = \sum_g a_g(x)\bar{g}$ where each a_g is a map from G to F. Equivalently,

$$\partial(\bar{x}) = \sum_g a_g(x)\bar{g}\bar{x}$$

and, of course, for each x only finitely many $a_g(x)$ can be nonzero.

Let $x, y, g \in G$ and consider the $\bar{g}\bar{x}\bar{y}$ terms in

$$t(x,y)\partial(\overline{xy}) = \partial(\bar{x}\bar{y}) = \partial(\bar{x})\bar{y} + \bar{x}\partial(\bar{y}).$$

Since G is abelian, we obtain

$$t(x,y)a_g(xy)\bar{g}\overline{xy} = a_g(x)\bar{g}\bar{x}\bar{y} + \bar{x}a_g(y)\bar{g}\bar{y}$$

and hence

$$a_g(xy)\bar{g}\bar{x}\bar{y} = a_g(x)\bar{g}\bar{x}\bar{y} + a_g(y)\lambda(x,g)\bar{g}\bar{x}\bar{y}.$$

In other words,

$$a_g(xy) = a_g(x) + a_g(y)\lambda(x,g)$$

for all $x, y, g \in G$.

In particular, if $g \in Z$, then $a_g(xy) = a_g(x) + a_g(y)$ and it follows that the map $\theta\colon G \to F^t[Z]^+$ given by

$$\theta(x) = \sum_{g \in Z} a_g(x)\bar{g}$$

is a group homomorphism. Thus θ determines a central derivation ∂_θ of $F^t[G]$.

On the other hand, suppose $g \notin Z$ and let $x, y \in G \setminus \mathbb{C}^t_G(g)$. Then

$$\begin{aligned} a_g(x) + a_g(y)\lambda(x,g) &= a_g(xy) \\ &= a_g(yx) = a_g(y) + a_g(x)\lambda(y,g) \end{aligned}$$

so we obtain

$$a_g(x)(1 - \lambda(y,g)) = a_g(y)(1 - \lambda(x,g)).$$

Since $\lambda(x,g)$ and $\lambda(y,g)$ are not equal to 1, it follows that

$$\frac{a_g(x)}{1 - \lambda(x,g)} = \frac{a_g(y)}{1 - \lambda(y,g)}$$

and thus these fractions depend only upon g. Writing $c_g \in F$ for this common value, we have $a_g(x) = c_g(1 - \lambda(x, g))$ for all $x \in G \setminus \mathbb{C}'_G(g)$. Furthermore, if $x, y \in G \setminus \mathbb{C}'_G(g)$, then

$$\begin{aligned} a_g(xy) &= a_g(x) + a_g(y)\lambda(x, g) \\ &= c_g(1 - \lambda(x, g)) + c_g(1 - \lambda(y, g))\lambda(x, g) \\ &= c_g(1 - \lambda(x, g)\lambda(y, g)) = c_g(1 - \lambda(xy, g)). \end{aligned}$$

But any element of $\mathbb{C}'_G(g)$ is a suitable product of two such elements of $G \setminus \mathbb{C}'_G(g)$, and therefore we conclude that

$$a_g(x) = c_g(1 - \lambda(x, g))$$

for all $x \in G$.

Now define $\gamma \in F'[\![G]\!]$ by

$$\gamma = \sum_{g \notin Z} c_g \bar{g}.$$

Then

$$[\gamma, \bar{x}] = \sum_{g \notin Z} c_g(1 - \lambda(x, g))\bar{g}\bar{x} = \sum_{g \notin Z} a_g(x)\bar{g}\bar{x}$$

and, since

$$\partial_\theta(\bar{x}) = \theta(x)\bar{x} = \sum_{g \in Z} a_g(x)\bar{g}\bar{x},$$

we conclude that $\partial(\bar{x}) = \partial_\theta(\bar{x}) + \mathrm{ad}_\gamma\, \bar{x}$ for all $x \in G$. Thus $\partial = \partial_\theta + \mathrm{ad}_\gamma$ is indeed the sum of a central and a pseudo-inner derivation. Finally, the uniqueness of this decomposition follows immediately from the fact that any central derivation maps $\bar{x}$ to $F'[Zx]$, while any pseudo-inner derivation sends $\bar{x}$ to the complementary subspace $F'[(G \setminus Z)x]$. ∎

For any F-algebra R, let $\mathrm{Der}_F\, R$ denote the Lie algebra of F-derivations of R. Then we have

COROLLARY 2.3. *Let $F'[G]$ be given with G a finitely generated commutative group. Then every F-derivation of $F'[G]$ is uniquely the sum of a central and an inner derivation. If in addition $\mathbb{Z}'(G) = 1$, then every F-derivation of $F'[G]$ is uniquely the sum of a scalar and an inner derivation. Furthermore, in this case,*

$$[\mathrm{Der}_F\, F'[G], \mathrm{Der}_F\, F'[G]] = \mathrm{InDer}\, F'[G]$$

is a simple Lie algebra.

Proof. The first part is an immediate consequence of the preceding theorem and Lemma 2.1(ii). Now suppose in addition that $\mathbb{Z}^t(G) = 1$, so that the central derivations of $F^t[G]$ are scalar. Since $\text{InDer}\, F^t[G] \triangleleft \text{Der}\, F^t[G]$ and since any two scalar derivations commute, we have

$$[\text{Der}_F\, F^t[G], \text{Der}_F\, F^t[G]] \subseteq \text{InDer}\, F^t[G]$$

Finally, $\text{InDer}\, F^t[G] \cong \mathscr{L}(G)$ is a simple Lie algebra, by Theorem 1.4, and $\mathscr{L}(G)$ is clearly nonabelian when $G \neq 1$. Thus the result follows. ∎

Next we consider $\text{Der}_F\, F^t[H]$ when H is a subsemigroup of G. As we will see, it is quite possible for pseudo-inner derivations of $F^t[G]$ to stabilize $F^t[H]$ and yet not be pseudo-inner on $F^t[H]$. In other words, there can exist elements $\gamma \in F^t[\![G]\!]$ with $\text{ad}_\gamma\, F^t[H] \subseteq F^t[H]$, but such that ad_γ does not agree with ad_δ on $F^t[H]$ for any $\delta \in F^t[\![H]\!]$. However, there is at least one case when this cannot occur.

LEMMA 2.4. *Let H be a subsemigroup of G and assume that*

$$\bigcap_{n=1}^{\infty} H_g^{-n} = \varnothing \qquad \text{for all } g \in G \setminus H.$$

Then any pseudo-inner derivation of $F^t[G]$ which stabilizes $F^t[H]$ determines a pseudo-inner derivation of $F^t[H]$.

Proof. Let $\gamma = \sum_x c_x \bar{x} \in F^t[\![G]\!]$ define a pseudo-inner derivation of $F^t[G]$ stabilizing $F^t[H]$. Fix $g \in \text{supp}\,\gamma$. The goal is to show that either $g \in H$ or $g \in \mathbb{C}_G^t(H)$. This will yield the result because, in the latter case, the $\bar{g}$-term of γ has no effect on the restriction of ad_γ to $F^t[H]$.

Let h be any element of H. Since

$$\text{ad}_\gamma \bar{h} = \sum_x c_x(1 - \lambda(h,x))\bar{x}\bar{h} \in F^t[H],$$

it follows that either $h \in \mathbb{C}_G^t(g)$ or $h \in g^{-1}H = Hg^{-1}$. Thus $H \subseteq \mathbb{C}_G^t(g) \cup Hg^{-1}$ and, by induction, we have $H \subseteq \mathbb{C}_G^t(g) \cup Hg^{-n}$ for all $n \geq 1$. Indeed, if the latter inclusion holds for some n, then

$$\begin{aligned} H &\subseteq \mathbb{C}_G^t(g) \cup Hg^{-n} \subseteq \mathbb{C}_G^t(g) \cup (\mathbb{C}_G^t(g) \cup Hg^{-1})g^{-n} \\ &= \mathbb{C}_G^t(g) \cup Hg^{-(n+1)} \end{aligned}$$

since $g \in \mathbb{C}_G^t(g)$. As a consequence,

$$H \subseteq \mathbb{C}_G^t(g) \cup \left(\bigcap_{n=1}^{\infty} Hg^{-n} \right).$$

Finally, if $g \notin H$, then $\bigcap_{n=1}^{\infty} Hg^{-n} = \varnothing$ by hypothesis. Then $H \subseteq \mathbb{C}_G^t(g)$ and the lemma is proved. ▮

COROLLARY 2.5. *Let $F^t[G]$ be given with $\mathbb{Z}^t(G) = 1$ and let H be a subsemigroup of G generating G as a group.*

(i) *Any F-derivation of $F^t[H]$ is uniquely the sum of a scalar derivation and the restriction of a pseudo-inner derivation of $F^t[G]$.*

(ii) *If $\bigcap_{n=1}^{\infty} Hg^{-n} = \varnothing$ for all $g \in G \setminus H$, then any F-derivation of $F^t[H]$ is uniquely the sum of a scalar and a pseudo-inner derivation.*

(iii) *If G is finitely generated, then the pseudo-inner derivations in* (i) *and* (ii) *are necessarily inner.*

Proof. Since $\mathbb{Z}^t(G) = 1$, Proposition 1.5 implies that any F-derivation of $F^t[H]$ extends uniquely to an F-derivation of $F^t[G]$. Part (i) now follows from Theorem 2.2 and part (ii) from the previous lemma. Of course, (iii) is a consequence of Lemma 2.1(ii). ▮

Interesting concrete examples of (ii) and (iii) above are contained in the following generalizations of [KPS, Theorem 1.2].

COROLLARY 2.6. *Let G be the free abelian group on the generators $\{x_i \mid i \in \mathscr{I}\}$. For some subset $\mathscr{J}$ of the index set $\mathscr{I}$, let $H = H_{\mathscr{J}}$ be the subsemigroup of G generated by all x_i and those x_j^{-1} with $j \in \mathscr{J}$. Suppose $F^t[G]$ is a twisted group algebra with $\mathbb{Z}^t(G) = 1$. Then any F-derivation of $F^t[H]$ is uniquely the sum of a scalar and a pseudo-inner derivation. Furthermore, if $\mathscr{I}$ is a finite set, then the preceding pseudo-inner derivation is necessarily inner.*

Proof. We need only show that H satisfies the hypothesis of part (ii) of the previous corollary. To this end, note that $H = H_{\mathscr{J}}$ is the set of all monomials in the generators, $x_i^{\pm 1}$ such that the exponent of x_i is nonnegative whenever $i \in \mathscr{I} \setminus \mathscr{J}$. Now suppose that $g \in G$ and that $\bigcap_{n=1}^{\infty} Hg^{-n} \neq \varnothing$. If w is a fixed element of this intersection, then $wg^n \in H$ for all $n \geq 1$, and it follows that the exponent of x_i in g must be nonnegative whenever $i \in \mathscr{I} \setminus \mathscr{J}$. Thus $g \in H$ and Corollary 2.5(ii)(iii) yields the result. ▮

We close this section with two examples which show that some assumption on the embedding of H in G is required for the conclusion of Lemma 2.4 to hold. In both cases, let $G = \langle x, y \rangle$ be free abelian on the two generators x, y and let $F^t[G]$ be the skew Laurent polynomial ring $F[\bar{x}, \bar{y}, \bar{x}^{-1}, \bar{y}^{-1}]$ with $\bar{x}\bar{y} = \lambda \bar{y}\bar{x}$ for some $\lambda \in F^{\cdot}$ of infinite multiplicative order. Then it is easy to see that $\mathbb{Z}^t(G) = 1$, so $F^t[G]$ is a simple ring.

Now let H_1 be the subsemigroup of G generated by x, y, z^2, and z^3 where $z = xy^{-1}$. Note that z^2 and z^3 generate all powers of z of degree

≥ 2, and therefore H_1 consists of all products $x^a y^b z^n$ with $a, b \geq 0$ and with $n = 0$ or $n \geq 2$. From this it is clear that $z \notin H_1$. On the other hand, it is easy to check that the product of z with any generator of H_1 is contained in H_1, and therefore $zH_1^{\#} \subseteq H_1$. As a consequence, the inner derivation $\mathrm{ad}_{\bar{z}}$ of $F^t[G]$ stabilizes $F^t[H_1]$. Furthermore, since $\mathbb{Z}^t(G) = 1$, the restriction of $\mathrm{ad}_{\bar{z}}$ to $F^t[H_1]$ is not an inner derivation. Thus the conclusion of Corollary 2.5(ii)(iii) does not apply to this subsemigroup.

As a second example, take H_2 to be the subsemigroup of G generated by y and all xy^{-n} with $n \geq 0$. Equivalently, H_2 is the set of all $x^a y^b$ such that $a \geq 1$ whenever $b < 0$. In particular, if $w = y^{-1}$, then $w \notin H_2$ and in fact no positive power of w is contained in H_2. Again, the product of w with any generator of H_2 is contained in H_2, so $wH_2^{\#} \subseteq H_2$. Thus $\mathrm{ad}_{\bar{w}}$ is an inner derivation of $F^t[G]$ which stabilizes $F^t[H_2]$, but $\mathrm{ad}_{\bar{w}}$ is not inner in its action on $F^t[H_2]$.

3. NONSCALAR AUTOMORPHISMS

In this section we consider the F-automorphisms of twisted group algebras $F^t[G]$ with G abelian. To start with, let $\alpha\colon G \to F^{\cdot}$ be a multiplicative homomorphism so that $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in G$. Then the F-linear operator θ_α on $F^t[G]$ given by $\theta_\alpha(\bar{x}) = \alpha(x)\bar{x}$ is easily seen to define an F-automorphism of the algebra. Indeed, if $x, y \in G$, then

$$\begin{aligned}\theta_\alpha(\bar{x}\bar{y}) &= \theta_\alpha(t(x,y)\overline{xy}) = t(x,y)\alpha(xy)\overline{xy}\\ &= \alpha(x)\bar{x}\alpha(y)\bar{y} = \theta_\alpha(\bar{x})\theta_\alpha(\bar{y}).\end{aligned}$$

We call any such θ_α obtained in this way a scalar automorphism of $F^t[G]$, and we let ScAut $F^t[G]$ denote the set of all such scalar automorphisms. Obviously,

$$\mathrm{ScAut}\, F^t[G] \cong \mathrm{Hom}(G, F^{\cdot}).$$

In the following, we will mainly be concerned with skew polynomial and skew Laurent polynomial rings. Thus it makes sense to assume that G is torsion free.

LEMMA 3.1. *If G is torsion-free abelian, then $\mathscr{A} = \mathrm{Aut}_F\, F^t[G]$ stabilizes the group $\mathscr{T}$ of trivial units of $F^t[G]$. Thus $\mathscr{S} = \mathrm{ScAut}\, F^t[G]$ is a normal subgroup of $\mathscr{A}$ with $\mathscr{A}/\mathscr{S}$ isomorphic to a subgroup of* Aut G.

Proof. Since G is torsion-free abelian, it is an ordered group, and hence all units of $F^t[G]$ are trivial. Thus $\mathscr{A} = \mathrm{Aut}_F\, F^t[G]$ stabilizes the group $\mathcal{G}$ of trivial units of $F^t[G]$, and of course $\mathcal{A}$ fixes $F^{\cdot} \triangleleft \mathcal{G}$ elementwise. As a consequence, $\mathcal{A}$ acts as automorphisms on $\mathcal{G}/F^{\cdot} \cong G$ and this yields a homomorphism χ: $\mathcal{A} \to \operatorname{Aut} G$. Certainly, $\mathcal{S} = \operatorname{ScAut}/F^t[G]$ is contained in the kernel of χ. Conversely, if $\theta \in \ker \chi$, then clearly $\theta(\bar{x}) = \alpha(x)\bar{x}$ for some function α: $G \to F^{\cdot}$. Furthermore, since θ is an F-automorphism, it follows easily that α is a multiplicative homomorphism. Thus $\theta = \theta_\alpha \in \mathcal{S}$, so $\mathcal{S} = \ker \chi$ and $\mathcal{A}/\mathcal{S}$ is isomorphic to a subgroup of Aut G. ∎

The goal now is to determine the nonscalar automorphisms of $F^t[G]$. In other words, we wish to find $\mathcal{A}/\mathcal{S}$ as a subgroup of Aut G. As it turns out, this is a fairly hopeless problem, but we indicate a few special cases where the computation can be done.

PROPOSITION 3.2. *Let $R = F[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n]$ be a skew polynomial ring with $\bar{x}_i\bar{x}_j = \lambda_{i,j}\bar{x}_j\bar{x}_i$ for all i, j, and assume that its corresponding skew Laurent polynomial ring S is simple. Then every F-automorphism θ of R is given by $\theta(\bar{x}_1) = k_i\bar{x}_{\sigma(i)}$, where $k_i \in F^{\cdot}$ and where σ is a permutation of the subscripts $\{1, 2, \dots, n\}$. Furthermore, every n-tuple $(k_1, k_2, \dots, k_n)$ of nonzero scalars can occur in this way, and the permutation σ occurs if and only if $\lambda_{i,j} = \lambda_{\sigma(i),\sigma(j)}$ for all i, j.*

Proof. If G is the free abelian group on the generators $x_1, x_2, \dots, x_n$, then obviously $S = F^t[G]$ is a suitable twisted group algebra. Furthermore, $R = F^t[H]$ where H is the subsemigroup of G generated by $x_1, x_2, \dots, x_n$. By assumption, S is simple and hence $\mathbb{Z}^t(G) = 1$ by Lemma 1.2. Thus, by Proposition 1.5, any F-automorphism of R extends uniquely to an F-automorphism of S. The previous lemma now implies that θ stabilizes the group $\mathcal{G}$ of trivial units of S, and therefore θ stabilizes $\mathcal{G} \cap F^t[H] = \mathcal{H} = \{a\bar{h} \mid a \in F^{\cdot}, h \in H\}$. In particular, θ permutes the atoms of $\mathcal{H}/F^{\cdot} \cong H$, and consequently θ is of the required form. Furthermore, by applying θ to the equation $\bar{x}_i\bar{x}_j = \lambda_{i,j}\bar{x}_j\bar{x}_i$, we see that $\lambda_{i,j} = \lambda_{\sigma(i),\sigma(j)}$ for all i, j.

Conversely, it is clear that if σ is as above and if $(k_1, k_2, \dots, k_n) \in (F^{\cdot})^n$ is any n-tuple, then $\theta(\bar{x}_i) = k_i\bar{x}_{\sigma(i)}$ determines an F-automorphism of R. ∎

We remark that the system of equations $\lambda_{i,j} = \lambda_{\sigma(i),\sigma(j)}$ given above can be described matrix theoretically. Indeed, if $[\delta_{i,\sigma(j)}]$ is the $n \times n$ permutation matrix corresponding to σ, then the system is easily seen to be equivalent to the matrix product

$$[\delta_{i,\sigma(j)}][\lambda_{i,j}][\delta_{i,\sigma(j)}]^T = [\lambda_{i,j}].$$

As usual, let $F^t[G]$ be given with G an abelian group. We recall that the function λ: $G \times G \to F^{\cdot}$ is defined by $\bar{x}\bar{y} = \lambda(x, y)\bar{y}\bar{x}$ for all $x, y \in G$.

Moreover, $\lambda(x, yz) = \lambda(x, y)\lambda(x, z)$ and $\lambda(y, x) = \lambda(x, y)^{-1}$ for all $x, y, z \in G$. For convenience, we let $\Lambda = \Lambda(G)$ be the subgroup of $F^{\cdot}$ generated by all such $\lambda(x, y)$.

LEMMA 3.3. *Let G be a free abelian group of rank n.*

(i) *$\Lambda = \Lambda(G)$ is a finitely generated abelian group with at most $n(n-1)/2$ generators.*

(ii) *If Λ_0 is the torsion subgroup of Λ and $|\Lambda_0| = k$, then $F^t[G^k]$ is a characteristic subalgebra of $F^t[G]$ with $\Lambda(G^k)$ torsion free. Furthermore, $F^t[G]$ is a free right and left $F^t[G^k]$-module of finite rank k^n, and $F^t[G]$ is simple if and only if $F^t[G^k]$ is simple.*

(iii) *Let $\sigma \in \operatorname{Aut} G$. Then σ lifts to an algebra automorphism of $F^t[G]$ if and only if $\lambda(x, y) = \lambda(\sigma(x), \sigma(y))$ for all $x, y \in G$.*

Proof. (i) If $G = \langle x_1, x_2, \ldots, x_n \rangle$, then the multiplicative properties of λ imply that the elements $\lambda(x_i, x_j)$ with $i < j$ generate Λ.

(ii) Since Λ is finitely generated, Λ_0 is indeed a subgroup of finite order. Let $|\Lambda_0| = k$ and let $G^k = \{g^k \mid g \in G\}$, so that G^k is a subgroup of G of index k^n. As a consequence, $F^t[G]$ is a free right and left $F^t[G^k]$-module of rank k^n. Furthermore, by Lemma 3.1, $F^t[G^k]$ is a characteristic subalgebra of $F^t[G]$. Next, observe that $\lambda(x^k, y^k) = \lambda(x, y)^{k^2} \in \Lambda^{k^2}$. Thus, since Λ^{k^2} is a torsion-free subgroup of Λ, we conclude that $\Lambda(G^k) \subseteq \Lambda^{k^2}$ is also torsion free. Finally, suppose $\mathbb{Z}^t(G) = 1$ and let $x \in \mathbb{Z}^t(G^k)$. Then $1 = \lambda(x, G^k) = \lambda(x^k, G)$, so $x^k = 1$ and hence $x = 1$. Conversely, suppose $\mathbb{Z}^t(G^k) = 1$ and let $y \in \mathbb{Z}^t(G)$. Then $y^k \in \mathbb{Z}^t(G^k)$, so $y^k = 1$ and again $y = 1$. Lemma 1.1(ii) now yields the result.

(iii) Let $\sigma \in \operatorname{Aut} G$. Suppose first that σ lifts to an F-automorphism θ of $F^t[G]$. Thus $\theta(\bar{x}) = \alpha(x)\overline{\sigma(x)}$ for some function $\alpha\colon G \to F^{\cdot}$, and by applying θ to the formula $\bar{x}\bar{y} = \lambda(x, y)\bar{y}\bar{x}$, we conclude immediately that $\lambda(x, y) = \lambda(\sigma(x), \sigma(y))$. Conversely, suppose that λ is a σ-stable map. Let $x_1, x_2, \ldots, x_n$ be a free generating set for G, and let $y_i = \sigma(x_i)$ for all i. Then $y_1, y_2, \ldots, y_n$ is also a free generating set and, by assumption, $\lambda_{i,j} = \lambda(x_i, x_j) = \lambda(y_i, y_j)$ for all i, j. Observe that $F^t[G]$ is the skew Laurent polynomial ring in $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ with $\bar{x}_i\bar{x}_j = \lambda_{i,j}\bar{x}_j\bar{x}_i$, and it is also the skew Laurent polynomial ring in $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n$ with relations $\bar{y}_i\bar{y}_j = \lambda_{i,j}\bar{y}_j\bar{y}_i$. Thus there exists an F-automorphism ϕ of $F^t[G]$ with $\phi(\bar{x}_i) = \bar{y}_i$ for all i, and ϕ is the required lifting of σ. ∎

In view of part (ii) above, it is crucial to study those twisted group algebras $F^t[G]$ with $\Lambda(G)$ torsion free. As a start, we offer an alternate interpretation of part (iii) of the preceding lemma. For convenience, let us write

$$\mathrm{NsAut}\, F^t[G] = \mathrm{Aut}_F\, F^t[G]/\mathrm{ScAut}\, F^t[G]$$

for the group of nonscalar automorphisms of $F^t[G]$. By Lemma 3.1, $\mathrm{NsAut} F^t[G]$ is isomorphic to a subgroup of $\mathrm{Aut}\, G$. Furthermore, since $\lambda(x, y) = \lambda(x^{-1}, y^{-1})$ for all $x, y \in G$, it follows from Lemma 3.3(iii) that $\mathrm{NsAut}\, F^t[G]$ always contains the inverse map on G.

THEOREM 3.4. *Let $F^t[G]$ be a twisted group algebra with G a free abelian group of rank n, and write*

$$\Lambda(G) = \langle f_1 \rangle \times \langle f_2 \rangle \times \cdots \times \langle f_s \rangle$$

as a finite direct product of the cyclic groups $\langle f_i \rangle \cong Z/m_i Z$. If

$$\lambda(x, y) = f_1^{e_1(x,y)} f_2^{e_2(x,y)} \cdots f_s^{e_s(x,y)} \qquad \text{for all } x, y \in G,$$

then each exponent map $e_i\colon G \times G \to Z/m_i Z$ is a bilinear skew-symmetric form on G (viewed additively), and $\mathbb{Z}^t(G) = \bigcap_i \mathrm{rad}\, e_i$. Furthermore, if $\mathrm{Sp}_n(Z, e_i)$ denotes the simplectic subgroup of $\mathrm{GL}_n(Z)$ which preserves the form e_i, then

$$\mathrm{NsAut}\, F^t[G] \cong \bigcap_i \mathrm{Sp}_n(Z, e_i).$$

Proof. The multiplicative properties of λ imply that each $e_i\colon G \times G \to Z/m_i Z$ is a bilinear skew-symmetric form on G (viewed additively). Furthermore, $g \in \mathbb{Z}^t(G)$ if and only if $\lambda(g, G) = 1$ and hence if and only if $e_i(g, G) = 0$ for all i. In other words, $\mathbb{Z}^t(G) = \bigcap_i \mathrm{rad}\, e_i$. Finally, note that $\mathrm{Aut}\, G \cong \mathrm{GL}_n(Z)$, the group of $n \times n$ invertible integer matrices. Furthermore, an automorphism σ of G stabilizes λ if and only if it preserves each form e_i. Thus Lemma 3.2(iii) yields the result. ∎

We close this section with a few special cases of interest. To start with, we assume that $\Lambda(G)$ is small and we extend [KPS, Theorem 1.5].

COROLLARY 3.5. *Let $F^t[G]$ be given with G free abelian of rank n.*

(i) *If $\Lambda(G) = 1$, then $F^t[G] \cong F[G]$ and* $\mathrm{NsAut}\, F^t[G] = \mathrm{Aut}\, G \cong GL_n(Z)$.

(ii) *If $\Lambda(G) = \langle f \rangle \cong Z/mZ$ is cyclic, then* $\mathrm{NsAut}\, F^t[G] \cong \mathrm{Sp}_n(Z, e)$ *where $e\colon G \times G \to Z/mZ$ is the skew-symmetric bilinear form associated with λ. Furthermore, if $F^t[G]$ is simple, then $m = 0$ and the form e is nonsingular. In particular, n must be an even integer.*

This is an immediate consequence of Theorem 3.4 and requires no additional proof. Note that if

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

then, by [N, Theorem IV.1], any nonsingular skew-symmetric Z-bilinear form is equivalent, via a change of basis, to the form determined by the block diagonal matrix

$$\mathrm{diag}(d_1 M, d_2 M, \ldots, d_t M).$$

Here the d_i are positive integers with $d_i \mid d_{i+1}$ for all i. Furthermore, different forms of the same degree can give rise to distinct symplectic subgroups of the general linear group. Next, we consider the other extreme where $\Lambda(G)$ is as large as possible. For convenience, we say that x is a generator of G if it is part of a free generating set.

COROLLARY 3.6. *Let $F^t[G]$ be a twisted group algebra with G free abelian of rank $n \geq 3$. If $\Lambda(G)$ is free abelian of rank $(n-1)/2$, then* NsAut $F^t[G] = \{1, \tau\}$ *where τ: $x \mapsto x^{-1}$ is the inverse map on G.*

Proof. Let x and y be generators of G and suppose that $\lambda(x, G) = \lambda(y, G)$. We claim that $y = x$ or x^{-1}. To this end, choose a free generating set $x_1, x_2, \ldots, x_n$ for G with $x = x_1$, and let $\bar{x}_i \bar{x}_j = \lambda_{i,j} \bar{x}_j \bar{x}_i$. Then $\Lambda(G)$ is generated by the $n(n-1)/2$ elements $\lambda_{i,j} \in F^{\cdot}$ with $i < j$. Thus, by assumption, these $\lambda_{i,j}$ must constitute a free generating set for $\Lambda = \Lambda(G)$. Observe that $\lambda(x_1, G) = \langle \lambda_{1,k} \mid k = 2, 3, \ldots, n \rangle$, and write $y = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Fix $i \geq 2$, and using $n \geq 3$ choose a subscript $j \neq 1, i$. Then $\lambda(y, G) = \lambda(x_1, G)$ contains the product $\lambda(y, x_j) = \prod_r \lambda_{r,j}^{a_r}$. But $\lambda_{i,j}^{a_i}$ is a factor here, and both subscripts are distinct from 1. Thus $i \neq j$ implies that $a_i = 0$ for all such i. In other words, $y = x_1^{a_1} = x^{a_1}$, and since y is a generator of G, we conclude that $a_1 = \pm 1$.

Now let $\sigma \in$ NsAut $F^t[G] \subseteq$ Aut G. If x is a generator of G, then so is $\sigma(x)$. Furthermore, Lemma 3.3(iii) implies that $\lambda(x, G) = \lambda(\sigma(x), G)$. Thus, by the result of the previous paragraph, we see that $\sigma(x) = x$ or x^{-1}. Consequently, every element of G is an eigenvector for σ with eigenvalue equal to ± 1, and hence σ is either the identity or the inverse map on G. ∎

Finally, we consider a few small values of n. If $n = 1$ then $\Lambda(G) = 1$, and if $n = 2$ then $\Lambda(G)$ is cyclic. Thus these cases are covered by Corollary 3.5. Now let $n = 3$ and suppose in addition that $\Lambda = \Lambda(G)$ is torsion free. In view of Corollary 3.5 we can assume that Λ is not cyclic, and Corollary 3.6 handles the case where Λ has three generators. The only other possibility is that $\Lambda(G) \cong Z \times Z$, and we discuss this case below.

PROPOSITION 3.7. *Let $F^t[G]$ be given with G free abelian of rank 3, and assume that $\Lambda(G) \cong Z \times Z$. Then G has a free generating set x_1, x_2, x_3 with $\lambda(x_1, x_2) = 1$, and* NsAut $F^t[G]$ *consists of those $\sigma \in$ Aut G with*

$$\sigma(x_1) = x_1^{\epsilon}, \qquad \sigma(x_2) = x_2^{\epsilon}, \qquad \sigma(x_3) = x_1^a x_2^b x_3^{\epsilon}$$

for all $a, b \in Z$ and $\varepsilon = \pm 1$.

Proof. Let $\Lambda(G) = \langle f_1 \rangle \times \langle f_2 \rangle$ and, as in Theorem 3.4, write

$$\lambda(x, y) = f_1^{e_1(x,y)} f_2^{e_2(x,y)} \qquad \text{for all } x, y \in G.$$

Since n is odd and e_1 is a skew-symmetric bilinear form, it follows that rad $e_1 = \{g \in G \mid e_1(g, G) = 0\}$ is a nontrivial pure subgroup of G. Thus there exists a generator $x_1 \in G$ with $e_1(x_1, G) = 0$. Write $G = \langle x_1 \rangle \times Y$, and consider the group homomorphism $\xi\colon Y \to Z$ given by $y \mapsto e_2(x_1, y)$. Since Y has rank 2, it follows that ker ξ is a nontrivial pure subgroup of Y. Thus there exists a free generating set x_1, x_2, x_3 of G with $e_2(x_1, x_2) = 0$. But $e_1(x_1, x_2)$ is also 0, so $\lambda(x_1, x_2) = 1$.

As usual, let $\bar{x}_i \bar{x}_j = \lambda_{i,j} \bar{x}_j \bar{x}_i$ for all i, j. Since $\lambda_{1,2} = 1$ and $\Lambda(G)$ is free abelian of rank 2, it follows that $\Lambda(G) = \langle \lambda_{1,3} \rangle \times \langle \lambda_{2,3} \rangle$. Again, we have

$$\lambda(x, y) = \lambda_{1,3}^{e'_1(x,y)} \lambda_{2,3}^{e'_2(x,y)} \qquad \text{for all } x, y \in G,$$

where e'_1 and e'_2 are suitable skew-symmetric bilinear forms. Note that rad $e'_1 = \langle x_2 \rangle$ and rad $e'_2 = \langle x_1 \rangle$.

Now let $\sigma \in$ NsAut $F^t[G]$. Since σ preserves both e'_1 and e'_2 by Theorem 3.4, it follows that σ must stabilize their respective radicals. Hence $\sigma(x_1) = x_1^{\epsilon_1}$, $\sigma(x_2) = x_2^{\epsilon_2}$, and of course $\sigma(x_3) = x_1^a x_2^b x_3^{\epsilon}$ for suitable integer exponents. But

$$\begin{aligned} \lambda_{1,3} = \lambda(x_1, x_3) &= \lambda(\sigma(x_1), \sigma(x_3)) \\ &= \lambda\left(x_1^{\epsilon_1}, x_1^a x_2^b x_3^{\epsilon}\right) = \lambda_{1,3}^{\epsilon_1 \epsilon}, \end{aligned}$$

so $\epsilon_1 \epsilon = 1$. Hence $\epsilon_1 = \epsilon = \pm 1$, and similarly $\epsilon_2 = \epsilon = \pm 1$. Conversely, if σ is as above with $\epsilon_1 = \epsilon_2 = \epsilon = \pm 1$, then σ preserves λ, and therefore $\sigma \in$ NsAut $F^t[G]$ by Lemma 3.3(iii). ∎

We remark that the group NsAut $F^t[G]$ above can be described as the set of all 3×3 matrices of the form

$$\begin{bmatrix} \epsilon & 0 & a \\ 0 & \epsilon & b \\ 0 & 0 & \epsilon \end{bmatrix}$$

with $a, b \in Z$ and $\epsilon = \pm 1$.

4. BLOCK ALGEBRAS AND DERIVATIONS

Our next task is to study a minor modification of the Block algebra as defined in [B]. As we will see in the next section, there is a close relation between these Lie algebras and the algebras $\mathscr{L}(G)$ considered previously.

Let G be an abelian multiplicative group, let F be a field, and suppose that e: $G \times G \to F$ is a skew-symmetric bilinear form. Specifically, this means that e satisfies

$$e(x, yz) = e(x, y) + e(x, z)$$
$$e(xy, z) = e(x, z) + e(y, z)$$
$$e(x, x) = 0$$

for all $x, y, z \in G$. Of course, we also have $e(x, y) = -e(y, x)$. Now let $F_e[G]$ be the ordinary group ring with F-basis G, and define an operation $[\ ,\ ]$: $F_e[G] \times F_e[G] \to F_e[G]$ linearly by

$$[x, y] = e(x, y)xy \qquad \text{for all } x, y \in G.$$

Notice that $[x, x] = 0$ and that

$$[[x, y], z] = e(x, y)e(xy, z)xyz = (e(x, y)e(y, z) \\ -e(z, x)e(x, y))xyz$$

for $x, y, z \in G$. Thus the Jacobi identity is satisfied, and $F_e[G]$ is a Lie algebra. Of course the associative multiplication in $F_e[G]$ is not at all related to the Lie structure here. Nevertheless, this multiplication exists and does come into play.

For convenience, we carry over the same notation used in the preceding study of twisted group algebras. Thus, for example, we will speak about the support of elements of $F_e[G]$, and if X is any subset of G, then we let

$$F_e[X] = \{\alpha \in F_e[G] \mid \operatorname{supp} \alpha \subseteq X\}.$$

Furthermore, we set

$$\mathbb{Z}_e(G) = \operatorname{rad} e = \{g \in G \mid e(g, G) = 0 = e(G, g)\}$$

and, for any $x \in G$, we write

$$\mathbb{C}_e(x) = \{g \in G \mid e(g, x) = 0 = e(x, g)\}.$$

These are of course both subgroups of G.

LEMMA 4.1. *Let $e: G \times G \to F$ be a skew-symmetric bilinear form and let $L = F_e[G]$. Then $\mathbb{Z}(L) = F_e[Z]$ where $Z = \mathbb{Z}_e(G)$, and $[L, L] = F_e[G \setminus Z]$. In particular, $L = \mathbb{Z}(L) \oplus [L, L]$ and $[L, L] \cong \operatorname{InDer} L$.*

Proof. Let $\alpha = \sum_x a_x x \in F_e[G] = L$. Then $\alpha \in \mathbb{Z}(L)$ if and only if

$$0 = [\alpha, g] = \sum_x a_x [x, g] = \sum_x a_x e(x, g) xg$$

for all $g \in G$, and hence if and only if $\operatorname{supp} \alpha \subseteq Z$.

Next let $x, y \in G$ with $xy \in Z$. Then $e(x, y) = e(x, xy) - e(x, x) = 0$, so $[x, y] = 0$. Consequently, no member of $[L, L]$ can have a supporting element in Z, and hence $[L, L] \subseteq F_e[G \setminus Z]$. Conversely, suppose $g \in G \setminus Z$, and choose $h \in G$ with $e(h, g) \neq 0$. Then we have

$$[h, h^{-1}g] = e(h, h^{-1}g)g = e(h, g)g$$

and therefore $Fg \subseteq [L, L]$. In other words, $[L, L] = F_e[G \setminus Z]$ and, since $F_e[G] = F_e[Z] \oplus F_e[G \setminus Z]$, the result follows. ∎

If $G^\#$ is the set of nonidentity elements of G, then the preceding lemma implies that $F_e[G^\#]$ is a Lie subalgebra of $F_e[G]$. We call this the Block algebra associated with the form e, and we denote it by $\mathscr{B}(G)$. Thus $\mathscr{B}(G)$ is a slightly simpler version of the algebra studied in [B], but it is not quite a special case. In particular, the following result is analogous to [B, Theorem 1], but is not a consequence of it.

THEOREM 4.2. *Let $\mathscr{B}(G)$ be the Block algebra associated with $e: G \times G \to F$, and assume that $|G| \geq 3$. Then $\mathscr{B}(G)$ is a simple Lie algebra if and only if $\mathbb{Z}_e(G) = 1$.*

Proof. Let $L = F_e[G]$, write $B = \mathscr{B}(G)$, and note that $\dim_F B \geq 2$ since $|G| \geq 3$. Thus, if B is simple, then $B = [B, B]$ and hence $\mathbb{Z}_e(G) = 1$ by Lemma 4.1.

Conversely, assume that $\mathbb{Z}_e(G) = 1$, and let I be a nonzero Lie ideal of B. If m is the minimal support size of the nonzero elements of I, then our first goal is to show that $m = 1$. To this end, let $\alpha = \sum_{i=1}^m a_i x_i$ be any element of I of support size m. If $g \in \mathbb{C}_e(x_1)$, then $[g, \alpha] \in I$ has support size less than m and therefore this element must be 0. Consequently, $e(g, x_i) = 0$ for all i and all such g. In particular, since $x_1 \in \mathbb{C}_e(x_1)$, it follows that $e(x_1, x_i) = 0$ for all i, and thus $e(x_i, x_j) = 0$ for all i, j. Now suppose that $g \in G \setminus \mathbb{C}_e(x_1)$. Then $[g, \alpha] \in I$ and

$$[g, \alpha] = \sum_{i=1}^{m} a_i e(g, x_i) g x_i$$

is nonzero, so this Lie commutator must have support size m. Therefore, by the above argument, the supporting elements of $[g, \alpha]$ satisfy

$$0 = e(gx_i, gx_j) = e(g,g) + e(g,x_j) + e(x_i,g) + e(x_i,x_j)$$
$$= e(g,x_j) - e(g,x_i) = e(g, x_j x_i^{-1})$$

for all i, j and all such g. But this equation also holds when $g \in \mathbb{C}_e(x_1)$, so we conclude that $e(G, x_j x_i^{-1}) = 0$. In other words, $x_j x_i^{-1} \in \mathbb{Z}_e(G) = 1$ and therefore $x_i = x_j$ for all i, j. Since the x_i are distinct, it follows that $m = 1$.

We now know that I contains Fz for some $z \in G^{\#}$. Furthermore, if $y \in G \setminus \mathbb{C}_e(z)$, then the formula for $[y, z]$ implies that I contains Fyz. But yz is a typical element of $X = G \setminus \mathbb{C}_e(z)$, so I contains $F_e[X]$. Finally, observe that X generates G as a group. In particular, if g is any element of $G^{\#}$, then $e(g, X)$ cannot equal 0. Thus there exists $x \in X$ with $g \notin \mathbb{C}_e(x)$, and the preceding argument now implies that $Fg \in I$. Consequently, $I = B$ and B is a simple Lie algebra. ∎

We remark that if $|G| = 2$, then $\dim_F \mathscr{B}(G) = 1$ so $\mathscr{B}(G)$ is Lie simple independent of the nature of e.

Again, let $e\colon G \times G \to F$ be a skew-symmetric bilinear form on the abelian group G and let $F_e[G]$ be its associated Lie algebra. Our next goal is to study the derivations of $F_e[G]$. For convenience, write $Z = \mathbb{Z}_e(G)$ so that $\mathbb{Z}(F_e[G]) = F_e[Z]$. As will be apparent, there are two types of F-derivations of $F_e[G]$ which are of particular interest; these are the central and the pseudo-inner derivations which we define below.

Let $\theta\colon G \to F_e[Z]^+$ be an almost linear map from the multiplicative group G to the additive group $F_e[Z]^+$. By this we mean that θ satisfies $\theta(xy) = \theta(x) + \theta(y)$ at least for those $x, y \in G$ with $e(x, y) \neq 0$. Obviously, this is equivalent to the assertion that

$$e(x,y)\theta(xy) = e(x,y)\theta(x) + e(x,y)\theta(y) \qquad \text{for all } x, y \in G.$$

Now given such a map θ, we define the F-linear operator $\partial = \partial_\theta$ by $\partial(x) = \theta(x)x$ for all $x \in G$. Then it is easy to see that ∂ is an F-derivation of $F_e[G]$. Indeed, let $x, y \in G$ and observe that $\theta(x) \in F_e[Z]$ implies that $e(g, y) = e(x, y)$ for all g in the support of $\partial(x) = \theta(x)x$. It follows that

$$[\partial(x), y] = [\theta(x)x, y] = e(x,y)\theta(x)xy,$$

and similarly $[x, \partial(y)] = e(x, y)\theta(y)xy$. Thus

$$[\partial(x), y] + [x, \partial(y)] = e(x,y)(\theta(x) + \theta(y))xy = e(x,y)\theta(xy)xy$$
$$= \partial(e(x,y)xy) = \partial([x,y])$$

and ∂ is a derivation, as claimed. Furthermore, note that $\partial(x) \in F_e[Zx]$ for all $x \in G$. We call any such ∂ obtained in this way a central derivation.

Next let $F_e[\![G]\!]$ denote the set of all possibly infinite formal sums $\gamma = \sum_g c_g g$ with $g \in G$ and $c_g \in F$. Then $F_e[\![G]\!]$ is no longer a Lie algebra, but it can be used to determine certain derivations of $F_e[G]$. Specifically, if γ is as given, then we can define $\mathrm{ad}_\gamma\colon F_e[G] \to F_e[\![G]\!]$ linearly by

$$\mathrm{ad}_\gamma x = \sum_g c_g[g, x] = \sum_g c_g e(g, x) gx$$

for all $x \in G$. Note that the Jacobi identity applied to each summand of γ yields

$$\mathrm{ad}_\gamma[x, y] = [\mathrm{ad}_\gamma x, y] + [x, \mathrm{ad}_\gamma y]$$

for all $x, y \in G$. In particular, if γ has the additional property that $\mathrm{ad}_\gamma x \in F_e[G]$ for all such x, then $\mathrm{ad}_\gamma\colon F_e[G] \to F_e[G]$ determines an F-derivation of the Lie algebra. We call any such derivation of $F_e[G]$ obtained in this manner a pseudo-inner derivation. Since the elements of $Z \cap \operatorname{supp} \gamma$ have no effect on the map ad_γ, we may always suppose that $Z \cap \operatorname{supp} \gamma = \varnothing$. Clearly, $\mathrm{ad}_\gamma x \in F_e[(G \setminus Z)x]$ for all $x \in G$. The following is the obvious analog of Lemma 2.1 and the proof is virtually identical.

LEMMA 4.3. *Let* $\gamma \in F_e[\![G]\!]$.

(i) *ad_γ is a pseudo-inner derivation if and only if*

$$(G \setminus \mathbb{C}_e(x)) \cap \operatorname{supp} \gamma$$

is finite for all $x \in G$.

(ii) *If G has finitely many elements $x_1, x_2, \ldots, x_n$ with $\bigcap_1^n \mathbb{C}_e(x_i) = \mathbb{Z}_e(G)$, then any pseudo-inner derivation of $F_e[G]$ is inner. In particular, this applies when G is a finitely generated group.*

In general, $F_e[G]$ can admit pseudo-inner derivations which are not inner. For example, let G be free abelian on the infinitely many generators $x_1, x_2, \ldots, y_1, y_2, \ldots$ and define the skew-symmetric form $e\colon G \times G \to F$ so that $e(x_i, y_j) = \delta_{i,j}$ and $e(x_i, x_j) = 0 = e(y_i, y_j)$ for all i, j. Then clearly $\mathbb{Z}_e(G) = 1$, so $\mathscr{B}(G)$ is simple. Furthermore, notice that if ∂ is any inner derivation of $F_e[G]$, then the elements y_i are eventually constant for ∂. Thus the element $\gamma = \sum_{i=1}^\infty x_i \in F_e[\![G]\!]$ determines a pseudo-inner derivation ad_γ which is not inner.

We will show in Theorem 4.5 why these derivations are of importance. But first we require the following technical result.

LEMMA 4.4. *Let g be a fixed element of $G \setminus \mathbb{Z}_e(G)$ and suppose that the function $f\colon G \to F$ satisfies the identity*

$$e(xg, y)f(x) + e(x, gy)f(y) = e(x, y)f(xy) \qquad \text{for all } x, y \in G.$$

If $f(v) = 0$ for some elements $v \in G \setminus \mathbb{C}_e(g)$, then $f(x) = 0$ for all $x \in G$.

Proof. First let $y = x^{-1}$ in the above identity. Since $e(g, x^{-1}) = -e(g, x) = e(x, g)$ and $e(x, y) = 0$, we obtain

$$e(x, g)f(x) + e(x, g)f(x^{-1}) = 0.$$

Thus when $e(x, g) \neq 0$, we have $f(x^{-1}) = -f(x)$. In particular, since $e(v, g) \neq 0$ and $f(v) = 0$, it follows that $f(v^{-1}) = 0$.

Now suppose, by way of contradiction, that $f(w) \neq 0$ for some $w \in G$. Then, setting $x = v$ and $y = w$ in the given identity yields

$$e(v, gw)f(w) = e(v, w)f(vw)$$

since $f(v) = 0$. Similarly, letting $x = v^{-1}$ and $y = vw$, we obtain

$$e(v, w)f(w) = e(v, gw)f(vw)$$

since $e(v^{-1}, gvw) = -e(v, gw)$, $e(v^{-1}, vw) = -e(v, w)$, and $f(v^{-1}) = 0$. Next, we subtract the latter displayed equation from the former. This yields

$$e(v, g)f(w) = -e(v, g)f(vw)$$

and, since $e(v, g) \neq 0$, it follows that $f(vw) = -f(w)$. Finally, by substituting this value for $f(vw)$, the previous subtrahend simplifies to

$$e(v, gw^2)f(w) = 0.$$

In other words, $f(w) \neq 0$ implies that $f(vw) \neq 0$ and that

$$0 = e(v, gw^2) = e(v, g) + 2e(v, w).$$

Note that $e(v, g) \neq 0$, so the latter equation implies that $2e(v, w) \neq 0$. In particular, F does not have characteristic 2, and

$$e(v, gw^{-2}) = e(v, g) - 2e(v, w) \neq 0.$$

The work of the previous paragraph now implies that $f(w^{-1}) = 0$. In particular, $f(w^{-1}) \neq -f(w)$, so the work of the first paragraph yields $e(w, g) = 0$. Finally, set $x = vw$ and $y = w^{-1}$ in the original identity. Since $f(w^{-1}) = f(v) = 0$, this yields

$$e(vwg, w^{-1})f(vw) = 0$$

and hence $0 = e(vwg, w^{-1}) = e(w, vg)$ since $f(vw) \neq 0$. But we have just shown that $e(w, g) = 0$, so $e(w, v) = e(w, vg) - e(w, g) = 0$ and this contradicts $2e(v, w) \neq 0$. Thus $f(w) = 0$, and the result follows. ∎

We can now quickly prove

THEOREM 4.5. *Let $F_e[G]$ be the Lie algebra associated with the skew-symmetric bilinear form $e\colon G \times G \to F$. Then any F-derivation of $F_e[G]$ is uniquely the sum of a central and a pseudo-inner derivation.*

Proof. Let ∂ be an F-derivation of $F_e[G]$ and, for each $x \in G$, write $\partial(x)x^{-1} = \sum_g a_g(x)g$ where each a_g is a map from G to F. Equivalently,

$$\partial(x) = \sum_g a_g(x)gx$$

and, of course, for each x only finitely many $a_g(x)$ can be nonzero.

Let $x, y, g \in G$ and consider the gxy coefficients in

$$[\partial(x), y] + [x, \partial(y)] = \partial([x, y]) = e(x, y)\partial(xy).$$

Since G is abelian, we obtain

$$e(xg, y)a_g(x) + e(x, gy)a_g(y) = e(x, y)a_g(xy),$$

the identity of the previous lemma.

Suppose first that $g \in \mathbb{Z}_e(G) = Z$. Then the above equation becomes

$$e(x, y)a_g(x) + e(x, y)a_g(y) = e(x, y)a_g(xy)$$

and it follows that the map $\theta\colon G \to F_e[Z]^+$ given by

$$\theta(x) = \sum_{g \in Z} a_g(x)g$$

is almost linear. Thus θ determines a central derivation ∂_θ of $F_e[G]$.

On the other hand, suppose $g \in G \setminus Z$ and choose an element $v_g \in G \setminus \mathbb{C}_e(g)$. Then $e(g, v_g) \neq 0$, so there exists a field element $c_g \in F$ with $a_g(v_g) = c_g e(g, v_g)$. Now define the function $f_g\colon G \to F$ by

$$f_g(x) = a_g(x) - c_g e(g, x) \qquad \text{for all } x \in G,$$

so that $f_g(v_g) = 0$. Since the identity for the function a_g can be rewritten as

$$e(g, y)a_g(x) - e(g, x)a_g(y) = e(x, y)\big(a_g(xy) - a_g(x) - a_g(y)\big),$$

it is trivial to see that this equation is also satisfied by the map $x \mapsto e(g, x)$ and hence by the function f_g. Thus the previous lemma applies to this situation, and we conclude that f_g is identically 0. In other words, $a_g(x) = c_g e(g, x)$ for all $x \in G$.

Now define $\gamma \in F_e[\![G]\!]$ by

$$\gamma = \sum_{g \notin Z} c_g g.$$

Then

$$\operatorname{ad}_\gamma x = \sum_{g \notin Z} c_g e(g, x) gx = \sum_{g \notin Z} a_g(x) gx$$

and, since

$$\partial_\theta(x) = \theta(x) x = \sum_{g \in Z} a_g(x) gx,$$

we conclude that $\partial(x) = \partial_\theta(x) + \operatorname{ad}_\gamma x$ for all $x \in G$. Thus $\partial = \partial_\theta + \operatorname{ad}_\gamma$ is indeed the sum of a central and a pseudo-inner derivation. Finally, the uniqueness of this decomposition follows immediately from the fact that any central derivation maps x to $F_e[Zx]$, while any pseudo-inner derivation sends x to the complementary subspace $F_e[(G \setminus Z)x]$. ∎

As a consequence of this and Lemma 4.3(ii) we have

COROLLARY 4.6. *Let $F_e[G]$ be the Lie algebra associated with the skew-symmetric bilinear form e: $G \times G \to F$. If G is finitely generated, then any F-derivation of $F_e[G]$ is uniquely the sum of a central and an inner derivation.*

Since any derivation of the Block algebra $\mathscr{B}(G)$ extends to a derivation of $F_e[G]$ by Lemma 4.1, the above two results also yield the structure of $\operatorname{Der}_F \mathscr{B}(G)$. Note that these results differ somewhat from [B, Theorem 2] in that psuedo-inner derivations which are not inner do not seem to arise in the latter context.

5. DEFORMATIONS

In this final section we consider two distinct topics. The first concerns almost linear maps like the ones used to describe the central derivations of $F_e[G]$. Specifically, let e: $G \times G \to F$ be a skew-symmetric bilinear form and let V be an additive abelian group. Then we say that the map θ: $G \to V$ is almost linear if $\theta(xy) = \theta(x) + \theta(y)$ at least for those $x, y \in G$ with $e(x, y) \neq 0$. Obviously, any group homomorphism is almost linear. Furthermore, if θ vanishes off rad e, then again it is an almost linear map. As we see below, these are the prototype examples at least when char $F \neq 2$. Note that, if char $F \neq 2$, then $G/\mathbb{C}_e(x) \subseteq F^+$ has no element of order 2, and hence $\mathbb{C}_e(x)$ cannot have index 2 in G.

THEOREM 5.1. *Suppose* $e: G \times G \to F$ *is a skew-symmetric bilinear form, and let* $\theta: G \to V$ *be an almost linear map. If no* $\mathbb{C}_e(x)$ *has index* 2 *in* G, *then* θ *is the sum of a group homomorphism and a function which vanishes off* rad e. *In particular, this applies when* char $F \neq 2$.

Proof. We can assume that $Z = \mathbb{Z}_e(G) = \text{rad } e$ is a proper subgroup of G. We will show that θ can be modified on Z in such a way that it becomes a group homomorphism on G. This will, of course, yield the result.

We first claim that $\theta(xy) = \theta(x) + \theta(y)$ whenever x, y, and xy are all elements outside Z. Since this equality follows directly from the definition of almost linear when $e(x, y) \neq 0$, we can assume in the course of this proof that $e(x, y) = 0$. Now observe that $\mathbb{C}_e(x)$, $\mathbb{C}_e(y)$, and $\mathbb{C}_e(xy)$ are proper subgroups of G which are not of index 2. Thus G properly contains $\mathbb{C}_e(x) \cup \mathbb{C}_e(y) \cup \mathbb{C}_e(xy)$, and we can choose an element $a \in G$ which is outside this union. Since $a \notin \mathbb{C}_e(x)$, we have $e(a, x) \neq 0$ and hence

$$\theta(a) + \theta(x) = \theta(ax).$$

Next, since $a \notin \mathbb{C}_e(y)$ and $e(x, y) = 0$, we have $e(ax, y) = e(a, y) \neq 0$ and hence

$$\theta(ax) + \theta(y) = \theta(axy).$$

Finally, $a \notin \mathbb{C}_e(xy)$ yields

$$\theta(axy) = \theta(a) + \theta(xy),$$

and by adding the above three displayed equations and canceling like terms, we obtain the required formula.

Next, we show that if $x, y, u, v \in G \setminus Z$ with $xy = uv \in Z$, then $\theta(x) + \theta(y) = \theta(u) + \theta(v)$. To this end, observe that $xy \in Z$ implies that $\mathbb{C}_e(x) = \mathbb{C}_e(y)$, and similarly $\mathbb{C}_e(u) = \mathbb{C}_e(v)$. In particular, $e(x, y) = 0$ and we can choose an element $a \in G$ with $a \notin \mathbb{C}_e(x) \cup \mathbb{C}_e(u)$. Now $a \notin \mathbb{C}_e(x)$ implies that

$$\theta(a) + \theta(x) = \theta(ax).$$

Furthermore, since $a \notin \mathbb{C}_e(x) = \mathbb{C}_e(y)$ and $e(x, y) = 0$, we have $e(ax, y) = e(a, y) \neq 0$ and hence

$$\theta(ax) + \theta(y) = \theta(axy).$$

By adding the previous two displayed equations and canceling the $\theta(ax)$ term, we obtain

$$\theta(x) + \theta(y) = \theta(axy) - \theta(a).$$

Similarly,

$$\theta(u) + \theta(v) = \theta(auv) - \theta(a),$$

and since $xy = uv$, this second claim is proved.

Recall that $Z \neq G$. Thus if z is any element of the radical Z, then we can write $z = xy$ with $x, y \in G \setminus Z$. Furthermore, for all such choices of x, y, the result of the preceding paragraph implies that the sum $\theta(x) + \theta(y)$ is always the same. Therefore, we can redefine $\theta(z)$ to equal the common value of this sum. Having done this, we now know from all the work we have done so far that

$$\theta(x) + \theta(y) = \theta(xy)$$

whenever $x, y \notin Z$.

It remains to verify this linearity condition when just one of x or y is in Z, and then when both are in Z. To start with, assume that $x \in Z$ and that $y \notin Z$, and choose $a \in G \setminus \mathbb{C}_e(y)$. Then $e(xa^{-1}, y)$, $e(a, y)$, and $e(a, ay)$ are all nonzero, so xa^{-1}, a, $ay \notin Z$. Thus, since $(xa^{-1})a = x \in Z$, we have $\theta(x) = \theta(xa^{-1}) + \theta(a)$, and therefore

$$\begin{aligned}\theta(x) + \theta(y) &= \theta(xa^{-1}) + \theta(a) + \theta(y)\\ &= \theta(xa^{-1}) + \theta(ay) = \theta(xa^{-1}ay) = \theta(xy),\end{aligned}$$

since $a, y \notin Z$ and xa^{-1}, $ay \notin Z$. Consequently, this case is proved.

Finally, suppose $x, y \in Z$ and write $x = uv$ with $u, v \notin Z$. Then

$$\theta(x) + \theta(y) = \theta(u) + \theta(v) + \theta(y) = \theta(u) + \theta(vy),$$

since $v \notin Z$. Furthermore, $u \notin Z$, so

$$\theta(u) + \theta(vy) = \theta(uvy) = \theta(xy)$$

and, with this, the theorem is proved. ∎

When it applies, the previous result obviously yields a more precise description of the central derivations of $F_e[G]$, and hence of all the F-derivations. Thus, it has numerous corollaries, but we only offer the following immediate consequence of Lemma 4.1 and Theorems 4.5 and 5.1.

COROLLARY 5.2. *Let $L = F_e[G]$ be given and assume that no $\mathbb{C}_e(x)$ has index 2 in G. Then any F-derivation of $[L, L] \cong$ InDer L is the sum of a pseudo-inner derivation and a central derivation determined by a group homomorphism $\theta\colon G \to F_e[Z]^+$. In particular, this applies when* char $F \neq 2$.

If $\mathbb{Z}_e(G) = 1$, then $[L, L] = \mathscr{B}(G)$. Thus the above describes the structure of $\mathrm{Der}_F\, \mathscr{B}(G)$ when the Block algebra is Lie simple.

We remark that the preceding two results are not true in general without the hypothesis on the subgroups $\mathbb{C}_e(x)$. For example, let $F = GF(2)$ and say that the multiplicative group X is a hyperbolic plane if X is elementary abelian of order 4, and $e(x, y) = 1$ for all distinct nonidentity elements x, y of X. Now let $G = A \times B$ be the elementary abelian group of order 16 with A and B hyperbolic planes and with $e(A, B) = 0$. In other words, if G is viewed additively as a 4-dimensional F-vector space, then e: $G \times G \to F$ is the nonsingular skew-symmetric bilinear form determined by the matrix

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We claim now that if $X \subseteq G$ is a hyperbolic plane, then either $X \cap A \neq 1$ or $X \cap B \neq 1$. Indeed, suppose by way of contradiction that these intersections are both equal to 1. Then, writing the three nonidentity elements of $X = \{1, x_1, x_2, x_3\}$ in terms of their A and B components, we must have $x_i = a_i b_i$ with the a_i distinct elements of $A^{\#} = A \setminus 1$ and with the b_i distinct members of $B^{\#}$. In particular,

$$e(x_1, x_2) = e(a_1 b_1, a_2 b_2) = e(a_1, a_2) + e(b_1, b_2) = 1 + 1 = 0,$$

and this contradicts the assumption on X.

Now define θ: $G \to F$ by

$$\theta(g) = \begin{cases} 0, & \text{if } g \in A \cup B \\ 1, & \text{otherwise} \end{cases}$$

Then certainly θ is not the sum of a group homomorphism and a function which vanishes off $\mathbb{Z}_e(G) = 1$. Nevertheless, we claim that θ is almost linear. To this end, observe that if $x, y \in G$ with $e(x, y) \neq 0$, then x, y, and xy are the three nonidentity elements of a hyperbolic plane X. Thus, since char $F = 2$, it follows that θ is almost linear if and only if the values of θ on any such $X^{\#}$ add to 0. Now, if $X = A$ or $X = B$, then the sum of the θ values is simply $0 + 0 + 0 = 0$. On the other hand, in the remaining cases, we know at least that $X \cap A \neq 1$ or $X \cap B \neq 1$. Moreover, since $e(A, B) = 0$, it is clear that both these intersections cannot be nontrivial. Thus X contains precisely one nonidentity element of $A \cup B$, and two nonidentity elements of $G \setminus (A \cup B)$, and the sum of the three θ values is $0 + 1 + 1 = 0$, as required.

Thus θ is almost linear, and it gives rise to a central derivation ∂_θ of $F_e[G]$. Indeed, according to the definition of θ and ∂_θ, the elements of $A \cup B$ are constants for this derivation, while the elements of $G \setminus (A \cup B)$ are fixed points.

Our final topic concerns an interesting relation between certain twisted group algebras $F^t[G]$ and the Lie algebra $F_e[G]$. In this situation, $F = C$ is the field of complex numbers, and of course G is an abelian multiplicative group. Note that, if $q \neq 1$ is a positive real number and if u is complex, then $q^u = 1$ and only if the real part of u is 0 and the complex part of u is an integer multiple of $2\pi/\ln q$. In particular, if $q^u = 1$ for a fixed u and uncountably many q, then $u = 0$.

Now let $c: G \times G \to C$ be an additive 2-cocycle, that is, a function satisfying

$$c(x,y) + c(xy,z) = c(x,yz) + c(y,z) \qquad \text{for all } x,y,z \in G.$$

Then, for any positive real number q, it follows that $t_q(x,y) = q^{c(x,y)}$ is a multiplicative 2-cycle on G, and hence t_q determines a twisted group algebra which we denote by $C^t[G]_q$. By definition, $C^t[G]_q$ has a basis $\bar{G}$ with

$$\bar{x}\bar{y} = t_q(x,y)\overline{xy} = q^{c(x,y)}\overline{xy} \qquad \text{for all } x,y \in G.$$

Furthermore, since G is abelian, we have

$$\bar{x}\bar{y} = \lambda_q(x,y)\bar{y}\bar{x} \qquad \text{for all } x,y \in G,$$

where

$$\lambda_q(x,y) = t_q(x,y)/t_q(y,x) = q^{c(x,y)-c(y,x)}.$$

For convenience, write $e(x,y) = c(x,y) - c(y,x)$, so that $e(x,y) = -e(y,x)$. Furthermore, since $\lambda_q(x,yz) = \lambda_q(x,y)\lambda_q(x,z)$, this multiplicative property of λ_q implies that

$$q^{e(x,yz)} = q^{e(x,y)+e(x,z)},$$

and since this equation holds for uncountably many q, we have

$$e(x,yz) = e(x,y) + e(x,z) \qquad \text{for all } x,y,z \in G.$$

Thus $e: G \times G \to C$ is a skew-symmetric bilinear form. Let $\mathscr{L}(G)_q$ be the Lie algebra $C^t[G^\#]_q$, and let $\mathscr{B}(G) = C_e[G^\#]$ be the Block algebra associated with e.

THEOREM 5.3. *Given the above assumptions and notation, we have:*

(i) *The Lie algebras $\mathscr{L}(G)_q$ with $q \neq 1$ are a deformation of $\mathscr{B}(G)$.*

(ii) *If $\mathscr{L}(G)_q$ is Lie simple for some q, then $\mathscr{B}(G)$ is a simple Lie algebra.*

(iii) *If $\mathscr{B}(G)$ is a simple Lie algebra and G is a countable group, then $\mathscr{L}(G)_q$ is Lie simple for almost all q.*

Proof. (i) Let $x, y \in G^{\#}$ and let $q \neq 1$ be a positive real number. Then, by working in $\mathscr{L}(G)_q \subseteq C^t[G]_q$, we have

$$[\bar{x}, \bar{y}] = \bar{x}\bar{y} - \bar{y}\bar{x} = (q^{c(x,y)} - q^{c(y,x)})\overline{xy}.$$

Consequently, if we change the basis by setting $x_q = \bar{x}/(q-1)$, then

$$[x_q, y_q] = \frac{q^{c(x,y)} - q^{c(y,x)}}{q-1}(xy)_q.$$

But

$$\lim_{q \to 1} \frac{q^{c(x,y)} - q^{c(y,x)}}{q-1} = c(x,y) - c(y,x) = e(x,y),$$

so it follows that $[x_q, y_q] \approx e(x,y)(xy)_q$ for $q \approx 1$. Since $[x,y] = e(x,y)xy$ in $\mathscr{B}(G)$, the Lie algebras $\mathscr{L}(G)_q$ are, by definition, a deformation of $\mathscr{B}(G)$.

(ii) Suppose that $\mathscr{L}(G)_q$ is a simple Lie algebra for some q. Then, as we observed in the first section, either $\mathbb{Z}^t(G)_q = 1$, or $|G| = 2$ and $\dim_C \mathscr{L}(G)_q = 1$. In the latter case, we have $\dim_C \mathscr{B}(G) = 1$, so $\mathscr{B}(G)$ is Lie simple. On the other hand, if $\mathbb{Z}^t(G)_q = 1$, then $\mathbb{Z}_e(G) = 1$. Indeed, if $g \in \mathbb{Z}_e(G)$, then $e(g, G) = 0$, so $\lambda_q(g, G) = q^{e(g,G)} = 1$ and $g \in \mathbb{Z}^t(G)_q = 1$. The result now follows from Theorem 4.2.

(iii) Conversely, suppose that G is a countable group and that $\mathscr{B}(G)$ is a simple Lie algebra. Then, by Theorem 4.2, either $\mathbb{Z}_e(G) = 1$, or $|G| = 2$ and $\dim_C \mathscr{B}(G) = 1$. In the latter case, we have $\dim_C \mathscr{L}(G)_q = 1$ and hence $\mathscr{L}(G)_q$ is Lie simple for all q. On the other hand, if $\mathbb{Z}_e(G) = 1$, then $\mathbb{Z}^t(G)_q = 1$ for almost all q. Indeed, suppose $\mathbb{Z}^t(G)_q \neq 1$ for some q, and let g be a nonidentity element of this subgroup. Then $g \notin \mathbb{Z}_e(G)$, so we can further choose $h \in G$ with $e(g,h) \neq 0$. Now $g \in \mathbb{Z}^t(G)_q$ implies that $1 = \lambda_q(g,h) = q^{e(g,h)}$, so the real part of $e(g,h)$ must be 0, and the imaginary part is nonzero and an integer multiple of $2\pi/\ln q$. Consequently, there are only countably many possibilities for q as a function of the pair g, h. But there are only countably many choices for g and h, so it follows that $\mathbb{Z}^t(G)_q = 1$ for almost all q, and Theorem 1.4 yields the result. ∎

This extends a remark in [KPS]. As we will see, part (iii) above is not true, in general, without the countability assumption. For this, we first need

LEMMA 5.4. *Suppose λ, μ: $G \to C^+$ are group homomorphisms. Then the map c: $G \times G \to C$ given by $c(x,y) = \lambda(x)\mu(y)$ for all $x, y \in G$ is an additive 2-cocycle.*

Proof. For $x, y, z \in G$, we have

$$\begin{aligned} c(x,y) + c(xy,z) &= \lambda(x)\mu(y) + \lambda(xy)\mu(z) \\ &= \lambda(x)\mu(y) + \lambda(x)\mu(z) + \lambda(y)\mu(z) \\ &= \lambda(x)\mu(yz) + \lambda(y)\mu(z) \\ &= c(x,yz) + c(y,z), \end{aligned}$$

and the result follows. ▮

As a consequence, we have

PROPOSITION 5.5. *Let G be an abelian group and let C be the complex field.*

(i) *Suppose $e\colon G \times G \to C$ is a skew-symmetric bilinear form. Then there exists an additive 2-cocycle $c\colon G \times G \to C$ with*

$$c(x,y) - c(y,x) = e(x,y) \qquad \text{for all } x, y \in G.$$

In particular, if $C_e[G]$ is given, then there exists a deformation $\mathscr{L}(G)_q$ of the Block algebra $\mathscr{B}(G) = C_e[G^{\#}]$.

(ii) *Suppose $C^t[G]$ is a twisted group algebra with G free abelian. Then, for any positive real number $q \neq 1$, there exists an additive 2-cocycle c: $G \times G \to C$ such that $C^t[G] = C^t[G]_q$. In other words, with an appropriate diagonal change of basis, the twisting function $t\colon G \times G \to C^{\cdot}$ satisfies*

$$t(x,y) = t_q(x,y) = q^{c(x,y)} \qquad \text{for all } x, y \in G.$$

In particular, the Lie algebra $\mathscr{L}(G)_q = C^t[G^{\#}]$ is a deformation of a suitable Block algebra.

Proof. (i) Assume that $e\colon G \times G \to C$ is given. The goal is to construct a suitable additive 2-cocycle c. To start with, it clearly suffices to consider $G/\mathbb{Z}_e(G)$, and thus we may assume that e is nonsingular. In particular, G is torsion free, and therefore G embeds in the rational vector space $Q \otimes G$. Since e clearly extends to a Q-bilinear form on $Q \otimes G$, we may now suppose that G is a Q-vector space with basis $\{x_i \mid i \in \mathscr{I}\}$. For each $i \in \mathscr{I}$, let $\mu_i\colon G \to Q^+ \subseteq C^+$ be the coordinate function corresponding to the basis element x_i. Then μ_i is a Q-linear functional and, for each $x \in G$, only finitely many $\mu_i(x)$ can be nonzero.

Linearly order the index set $\mathscr{I}$, and define

$$c(x,y) = \sum_{a<b} e(x_a, x_b)\mu_a(x)\mu_b(y) \qquad \text{for all } x, y \in G.$$

Then it follows from the previous lemma and the comments of the first paragraph that $c\colon G \times G \to C$ is an additive 2-cocycle with $c(x_i, x_i) =$

$0 = e(x_i, x_i)$. Furthermore, if $i < j$, then $c(x_i, x_j) = e(x_i, x_j)$ and $c(x_j, x_i) = 0$. Thus

$$c(x_i, x_j) - c(x_j, x_i) = e(x_i, x_j)$$

and

$$c(x_j, x_i) - c(x_i, x_j) = -e(x_i, x_j) = e(x_j, x_i).$$

Since the maps $e(x, y)$ and $c(x, y) - c(y, x)$ are Q-bilinear functions which agree on a basis, it follows that they must be identical.

(ii) Let $q \neq 1$ be a fixed positive real number. If $\{x_i \mid i \in \mathscr{I}\}$ is a free generating set for G, then $C^t[G]$ is the skew Laurent polynomial ring in the variables $\bar{x}_i^{\pm 1}$ with relations $\bar{x}_i\bar{x}_j = \lambda_{i,j}\bar{x}_j\bar{x}_i$. Of course, $\lambda_{i,i} = 1$ and $\lambda_{i,j} = \lambda_{j,i}^{-1}$. Linearly order the index set $\mathscr{I}$ and, for each $i < j$, choose a complex number $e_{i,j}$ with $\lambda_{i,j} = q^{e_{i,j}}$. In addition, set $e_{j,i} = -e_{i,j}$ and $e_{i,i} = 0$. Then clearly $\lambda_{i,j} = q^{e_{i,j}}$ for all i, j. Now define the skew-symmetric bilinear form $e\colon G \times G \to C$ by $e(x_i, x_j) = e_{i,j}$ for all i, j. Then part (i) above implies that $\mathscr{B}(G) = C_e[G^\#]$ has a deformation $\mathscr{L}(G)_q = C^t[G^\#]_q$. Furthermore, note that $C^t[G]_q$ is also a skew Laurent polynomial ring in the variables $\bar{x}_i^{\pm 1}$, and that these new variables satisfy

$$\bar{x}_i\bar{x}_j = q^{e(x_i, x_j)}\bar{x}_j\bar{x}_i = q^{e_{i,j}}\bar{x}_j\bar{x}_i = \lambda_{i,j}\bar{x}_j\bar{x}_i.$$

Thus $C^t[G] = C^t[G]_q$, as required. ∎

Finally, we construct an uncountable example to show that Theorem 5.3(iii) requires the countability hypothesis. Let $\mathscr{I}$ be the set of positive real numbers different from 1, and let G be the free abelian group on the free generating set $\{x_i \mid i \in \mathscr{I}\}$. Define $e\colon G \times G \to C$ to be the skew-symmetric bilinear form determined by

$$e(x_i, x_j) = \begin{cases} 2\pi\sqrt{-1}/\ln i, & \text{if } ij = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then, by part (i) of the preceding proposition, $\mathscr{B}(G) = C_e[G^\#]$ has a deformation $\mathscr{L}(G)_q = C^t[G^\#]_q$, where q runs through the positive real numbers. Note that, if $q = 1$, then $C^t[G]_q \cong C[G]$ and $\mathbb{Z}^t(G)_q = G \neq 1$. In addition, if $q \neq 1$, then $q \in \mathscr{I}$ and we have

$$t_q(x_q, x_j) = q^{e(x_q, x_j)} = 1 \qquad \text{for all } j \in \mathscr{I}$$

since $q^{2\pi\sqrt{-1}/\ln q} = 1$. Thus $x_q \in \mathbb{Z}^t(G)_q$ here, and it follows that $\mathbb{Z}^t(G)_q \neq 1$ for all positive real numbers q. Consequently, $\mathscr{L}(G)_q$ is never Lie simple. On the other hand, e is clearly nonsingular, so $\mathscr{B}(G)$ is Lie simple by Theorem 4.2.

REFERENCES

[B] R. Block, On torsion-free abelian groups and Lie algebras, *Proc. Amer. Math. Soc.* **9** (1958), 613–620.

[KPS] E. Kirkman, C. Procesi, and L. Small, *A q-analog for the Virasoro algebra*, to appear.

[MPe] J. C. McConnell and J. J. Petit, *Crossed products and multiplicative analogues of Weyl algebras*, J. London Math. Soc. (2) **38** (1988), 47–55.

[N] M. Newman, "Integral Matrices," Academic Press, New York, 1972.

[Pa] D. S. Passman, "Infinite Crossed Products," Academic Press, Boston, 1989.