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Delta Ideals of Lie Color Algebras

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Let $L = \bigoplus_{g \in G} L_g$ be a Lie color algebra (possibly restricted) over the field K and graded by the finite abelian group G. If $\Delta_\infty(L) = \{l \in L \mid \dim_K[l, L] \text{ is countable}\}$, then $\Delta_\infty(L)$ is the (restricted) Lie color ideal of L generated by all (restricted) countable-dimensional Lie color ideals of L. We use $\Delta_\infty(L)$ to examine the symmetric Martindale quotient ring of the enveloping algebra $U(L)$ (or the restricted enveloping algebra when char $K = p > 0$). Specifically, we prove

THEOREM. *If $\Delta_\infty(L) = 0$, then $U(L)$ is symmetrically closed.*

We also examine the Lie color ideal $\Delta(L) = \{l \in L \mid \dim_K[l, L] \text{ is finite}\}$ and the possibly smaller ideal Δ_L, which is the join of all finite-dimensional Lie color ideals of L. Note that $\Delta(L) = \Delta_L$ when char $K = p > 0$, but that $\Delta(L)$ can be considerably larger than Δ_L when char $K = 0$. Nevertheless, we prove

THEOREM. $[\Delta(L), \Delta(L)] \subseteq \Delta_L$.

We remark that these results are new and of interest even when L is an ordinary or super Lie algebra. In fact, we consider Lie color algebras here only because we can obtain the more general facts with little additional work.

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1. INTRODUCTION AND TERMINOLOGY

In this paper, we study the symmetric Martindale quotient ring of the enveloping algebra (or restricted enveloping algebra) $U(L)$ of the Lie color algebra L. Specifically, in Section 2, we prove

THEOREM. *If $\Delta_x(L) = 0$, then $U(L)$ is symmetrically closed.*

In addition, we examine the relationship between the Lie color ideals $\Delta(L)$ and Δ_L, which were used in [BP1], [BP2], [BP3], [BP4], [W] to study various ring-theoretic properties of $U(L)$ when L is infinite-dimensional. Although $\Delta(L)$ can be considerably larger than Δ_L when the field has characteristic 0, we show in Section 3 that

THEOREM. $[\Delta(L), \Delta(L)] \subseteq \Delta_L$.

At this point, it is appropriate to define the concepts which will be used throughout the remainder of this paper. To start with, let L be a vector space over a field K and let G be a group. Then L is a *G-graded algebra* if there exist K-subspaces L_g such that $L = \oplus_{g \in G} L_g$ and a K-linear multiplication $[\ ,\ ]$: $L \times L \to L$ satisfying $[L_g, L_h] \subseteq L_{gh}$ for all $g, h \in G$. As usual, the elements of $\bigcup_{g \in G} L_g$ are said to be *homogeneous*. If G is a finite abelian group, then we call a map ϵ: $G \times G \to K^*$ a *bicharacter* if $\epsilon(g, hk) = \epsilon(g, h)\epsilon(g, k)$ and $\epsilon(g, h) = \epsilon(h, g)^{-1}$ for all $g, h, k \in G$.

DEFINITION. We say that L is a Lie color algebra over a field K if L is a G-graded algebra and there exists a bicharacter ϵ: $G \times G \to K^*$ such that $[x, y] = -\epsilon(g, h)[y, x]$ and $[[x, y], z] = [x, [y, z]] - \epsilon(g, h)[y, [x, z]]$ for all $x \in L_g$, $y \in L_h$, and $z \in L$. (Minor additional assumptions are required if char $K = 2$ or 3.)

Let G, ϵ, and L be as above. If $g \in G$, then it is easy to check that $\epsilon(g, g) = 1$ or -1. Thus, we can partition G into the sets $G_+ = \{g \in G \mid \epsilon(g, g) = 1\}$ and $G_- = \{g \in G \mid \epsilon(g, g) = -1\}$, and we define $L_+ = \oplus_{g \in G_+} L_g$ and $L_- = \oplus_{g \in G_-} L_g$. If char $K = 2$, then by convention $G = G_+$.

Given a Lie color algebra L, the Poincaré–Birkhoff–Witt theorem guarantees that there exists a unique largest K-algebra $U(L)$ containing L such that $U(L)$ is generated by L with relations $xy - \epsilon(g, h)yx = [x, y]$, for all $x \in L_g$ and $y \in L_h$. More precisely, we have

POINCARÉ–BIRKHOFF–WITT THEOREM. *Let $\mathcal{B}$ be a totally ordered basis for L consisting of homogeneous elements. Then $U(L)$ has as a K-basis the collection of all ordered monomials $b_1^{\beta_1} b_2^{\beta_2} \dots b_n^{\beta_n}$, such that $b_i \in \mathcal{B}$, $b_1 < b_2 < \cdots < b_n$, β_i is a nonnegative integer, and $\beta_i \leq 1$ whenever $b_i \in L_-$.*

If char $K = p > 0$, there is an additional structure which comes into play.

DEFINITION. We say that L is a restricted Lie color algebra over a field K of characteristic $p > 0$ if L is a Lie color algebra with a pth power map ${}^{[p]}: L_+ \to L_+$ satisfying

(i) $(\alpha x)^{[p]} = \alpha^p x^{[p]}$, for all $\alpha \in K$ and $x \in L_+$,

(ii) $[x^{[p]}, y] = (\mathrm{ad}_x)^p(y)$, for all $x \in L_+$ and $y \in L$,

(iii) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, for all $x, y \in L_+$.

Here $\mathrm{ad}_x(y) = [x, y]$ and $is_i(x, y)$ is the coefficient of λ^{i-1} in $(\mathrm{ad}_{\lambda x + y})^{p-1}(x)$.

If L is a restricted Lie color algebra, then there exists a unique largest K-algebra $U(L)$ containing L such that $U(L)$ is generated by L with the relations $xy - \epsilon(g,h)yx = [x, y]$, for all $x \in L_g$ and $y \in L_h$, and with $x^p = x^{[p]}$, for all $x \in L_+$. The obvious analog of Jacobson's theorem on restricted enveloping algebras implies that $U(L)$ exists (see [BMPZ]). Indeed we have

JABOSCON'S THEOREM. *Let L be a restricted Lie color algebra over a field K of characteristic $p > 0$ and let $\mathscr{B}$ be a totally ordered basis for L consisting of homogeneous elements. Then $U(L)$ has as a K-basis the set of all ordered monomials $b_1^{\beta_1} b_2^{\beta_2} \dots b_n^{\beta_n}$, such that $b_i \in \mathscr{B}$, $b_1 < b_2 < \cdots < b_n$, $0 \leq \beta_i < p$ whenever $b_i \in L_+$, and $0 \leq \beta_i \leq 1$ whenever $b_i \in L_-$.*

Regardless of the characteristic of K, we will refer to $U(L)$ as the *enveloping algebra* of L with the following understanding. If $\operatorname{char} K = 0$, then $U(L)$ is the algebra with K-basis described by the Poincaré–Birkhoff–Witt theorem. On the other hand, if $\operatorname{char} K = p > 0$, then we assume that L is restricted and that $U(L)$ has K-basis described by Jacobson's theorem. We will refer to the basis monomials which occur in the statement of the Poincaré–Birkhoff–Witt theorem or of Jacobson's theorem as *straightened monomials*.

If X is any subset of L, we let X^n denote the K-linear span in $U(L)$ of all products of the form $x_1 x_2 \dots x_k$, where $x_i \in X$ and $0 \leq k \leq n$, and we set $X^\infty = \bigcup_{n=0}^\infty X^n$. Furthermore, we say that X is *homogeneous* if it contains the homogeneous components of all of its elements, or equivalently if $X \subseteq \bigoplus_{g \in G}(X \cap L_g)$. Note that $U(L) = L^\infty$ and that if X is homogeneous then $X^1 L^1 = L^1 X^1$. Of course, the grading of L by G extends to a grading of $U(L)$ and we refer to homogeneous elements and homogeneous subsets of $U(L)$ in a completely analogous manner.

If $0 \neq \alpha \in U(L)$, then the degree of α is defined to be the minimal m with $\alpha \in L^m$. From the Poincaré–Birkhoff–Witt theorem, it follows that if $b_1^{\beta_1} b_2^{\beta_2} \dots b_n^{\beta_n}$ is a straightened monomial, then its degree is equal to

$\beta_1 + \beta_2 + \cdots + + \beta_n$. Furthermore, if α is given by

$$\alpha = \sum k_\beta b_1^{\beta_1} b_2^{\beta_2} \ldots b_n^{\beta_n},$$

then $\deg \alpha = \max\{\beta_1 + \beta_2 + \cdots + \beta_n \mid k_\beta \neq 0\}$.

The following observation is an immediate consequence of the Poincaré–Birkhoff–Witt theorem or of Jacobson's theorem.

LEMMA 1.1. *Let L be a Lie color algebra and let H be a homogeneous subalgebra of L. If $\mathcal{Y}$ is an ordered homogeneous basis for H and if $\mathcal{Y} \dot\cup \mathcal{X}$ is an ordered homogeneous basis for L with $\mathcal{X} < \mathcal{Y}$, then every element α of $U(L)$ can be uniquely expressed as a finite sum $\alpha = \sum_\nu \nu\alpha_\nu$, where ν is a straightened monomial in the elements of $\mathcal{X}$ and $\alpha_\nu \in U(H)$. Similarly, if $\mathcal{X} > \mathcal{Y}$, then every element $\alpha \in U(L)$ can be uniquely expressed as $\alpha = \sum_\nu \alpha'_\nu \nu$, again with $\alpha'_\nu \in U(H)$.*

In either of the above situations, we say that α is written *based on H*. We now move on to certain ideals and subspaces of L. To start with, let

$$\Delta(L) = \{l \in L \mid \dim_K[l, L] \text{ is finite}\},$$

$$\Delta_L = \{l \in L \mid l \text{ is contained in a finite-dimensional Lie color ideal of } L\},$$

and

$$\Delta_\infty(L) = \{l \in L \mid \dim_K[l, L] \text{ is countable}\}.$$

Furthermore, for any $g \in G$, let

$$\mathbb{D}_\infty(L_g) = \{l \in L \mid \dim_K[l, L_g] \text{ is countable}\}.$$

Basic properties of $\Delta(L)$ and Δ_L can be found in [BP3], [BP4], [W]. We conclude this section with some basic properties of $\Delta_\infty(L)$ and $\mathbb{D}_\infty(L_g)$.

LEMMA 1.2. *With the above notation, we have*

(i) *$\mathbb{D}_\infty(L_g)$ and $\Delta_\infty(L) = \bigcap_{g \in G} \mathbb{D}_\infty(L_g)$ are homogeneous subspaces of L.*

(ii) *If A is a finite subset of $\mathbb{D}_\infty(L_g)$, for some $g \in G$, then there exists a subspace M_g of countable codimension in L_g such that $[A, M_g] = 0$. Furthermore, if A is homogeneous then $[M_g, A] = 0$.*

(iii) *If A is a finite subset of $\Delta_\infty(L)$, then there exists a homogeneous subspace M of countable codimension in L such that $[A, M] = 0$. Furthermore, if A is homogeneous, then $[M, A] = 0$.*

(iv) *$\Delta_\infty(L)$ is a (restricted) Lie color ideal of L. In fact, it is the union of all the countable dimensional (restricted) Lie color ideals of L.*

Proof.

(i) If $x = \sum_{h \in G} x_h \in L$ is written in terms of its homogeneous components and if $y \in L_g$, then $[x_h, y]$ is easily seen to be the hg-component of $[x, y]$. It follows that $[x_h, L_g]$ is the image of $[x, L_g]$ under the natural linear transformation $L \to L_{hg}$, and hence $\dim_K[x_h, L_g] \leq \dim_K[x, L_g]$. In particular, if $x \in \mathbb{D}_\infty(L_g)$, then $\dim_K[x, L_g]$ is countable and therefore so is $\dim_K[x_h, L_g]$. Thus $\mathbb{D}_\infty(L_g)$ is homogeneous and, since G is finite, it follows that $\Delta_\infty(L) = \bigcap_{g \in G} \mathbb{D}_\infty(L_g)$ is also homogeneous.

(ii) Suppose $A = \{a_1, \ldots, a_n\} \subseteq \mathbb{D}_\infty(L_g)$ and observe that the linear transformation $L_g \to L \oplus L \oplus \cdots \oplus L$ given by $l \mapsto [a_1, l] \oplus [a_2, l] \oplus \cdots \oplus [a_n, l]$ has a countable-dimensional image. Thus the kernel M_g has countable codimension in L_g and satisfies $[A, M_g] = 0$. Furthermore, if A is homogeneous, then clearly $[M_g, A] = [A, M_g] = 0$ since M_g is also homogeneous.

(iii) Since $A \subseteq \Delta_\infty(L) = \bigcap_{g \in G} \mathbb{D}_\infty(L_g)$, we can take $M = \oplus_{g \in G} M_g$, where each M_g is given by (ii).

(iv) We first show that $\Delta_\infty(L)$ is a (restricted) Lie color ideal. To this end, let $x \in \Delta_\infty(L)$ and let $l \in L$ both be homogeneous. Then

$$[[x, l], L] \subseteq [x, [l, L]] + [[x, L], l] \subseteq [x, L] + [[x, L], l].$$

Hence, since $\dim_K[x, L]$ is countable, so is $\dim_K[[x, l], L]$ and $[x, l] \in \Delta_\infty(L)$. In the restricted case, if $x \in L_+$, then $[x^{[p]}, l] = (\mathrm{ad}_x)^p(l)$ so $[x^{[p]}, L] \subseteq [x, L]$ has countable dimension and $x^{[p]} \in \Delta_\infty(L)$. Since $\Delta_\infty(L)$ is homogeneous, this fact is proved.

Now if Y is a countable-dimensional Lie ideal of L, then $[Y, L] \subseteq Y$ implies that $Y \subseteq \Delta_\infty(L)$. Conversely, suppose $z \in \Delta_\infty(L)$ is homogeneous, set $Z_0 = Kz$, and inductively define

$$Z_{n+1} = [Z_n, L] = [Z_0, L, L, \ldots, L].$$

Since $\Delta_\infty(L)$ is an ideal, each Z_n is contained in $\Delta_\infty(L)$ and inductively each Z_n has countable dimension. Thus $Z = \bigcup_{n=0}^{\infty} Z_n$ is a countable-dimensional homogeneous Lie color ideal of L with $z \in Z \subseteq \Delta_\infty(L)$. In the restricted case, take $Z_{n+1} = K(Z_n)_+^{[p]} + [Z_n, L] \subseteq \Delta_\infty(L)$. Again, if $\dim_K Z_n$ is countable, then so is $\dim_K[Z_n, L]$. Furthermore, modulo $[Z_n, L]$, the space $K(Z_n)_+^{[p]}$ is spanned by the images of a basis for $K(Z_n)_+$ under the pth power map. Thus $\dim_K Z_{n+1}$ is also countable and again we take $Z = \bigcup_{n=0}^{\infty} Z_n$. ∎

In view of (iv) above, any simple Lie color algebra L of uncountable dimension satisfies $\Delta_\infty(L) = 0$. For example, we could take K to be an uncountable field of characteristic 0 and let L be the Virasoro algebra with K-basis $\{e_\alpha \mid \alpha \in K\}$ and relations $[e_\alpha, e_\beta] = (\alpha - \beta)e_{\alpha+\beta}$ for all $\alpha, \beta \in K$.

2. SYMMETRIC QUOTIENT RINGS

If L is a Lie color algebra over a field K with $\Delta_L = 0$ then it follows from [W, Theorem 5.3] that $U(L)$ is a prime ring. In this case, $U(L)$ has an interesting overring $Q_s(U(L))$, its *symmetric Martindale ring of quotients.* Basic properties of this K-algebra can be found in [P, Proposition 1.4]. In particular, if $q \in Q_s(U(L))$, then there exists a nonzero ideal $I \subseteq U(L)$ with $Iq, qI \subseteq U(L)$. Furthermore, if J is any nonzero ideal of $U(L)$ with Jq or qJ equal to 0, then $q = 0$. If $U(L) = Q_s(U(L))$, then we say that $U(L)$ is *symmetrically closed.* In this section, we show that $\Delta_x(L) = 0$ implies that $U(L)$ is symmetrically closed.

An example is given in [BP4] of a Lie superalgebra L over a field of characteristic 0 with $\Delta_L = 0$, but with $U(L)$ not symmetrically closed. Furthermore, [BP1, Example 7.6] exhibits a restricted Lie algebra L such that $\Delta_L = \Delta(L) = 0$, and yet $U(L)$ is not symmetrically closed. In light of these examples, it is not unreasonable to consider the stronger assumption $\Delta_x(L) = 0$.

If $l \in L_g$, then the operator ad_l extends to a linear map ${}^{\partial(l)}\colon U(L) \to U(L)$ defined on homogeneous elements by

$$a^{\partial(l)} = la - \epsilon(g,h)al,$$

for all $a \in U(L)_h$. In addition, $\partial(l)$ has the property that

$$(ab)^{\partial(l)} = a^{\partial(l)}b + \epsilon(g,h)ab^{\partial(l)},$$

for all $a \in U(L)_h$ and $b \in U(L)$. If $\delta = x_1^{m_1}x_2^{m_2}\dots x_r^{m_r}$ is a monomial in $U(L)$, with each x_i homogeneous, then for any $\xi \in U(L)$, we let ξ^δ denote the image of ξ under the composition $\partial(x_1)^{m_1}\,\partial(x_2)^{m_2}\cdots\partial(x_r)^{m_r}$. We note that if ξ is homogeneous and belongs to an ideal I of $U(L)$, then $\xi^\delta \in I$. However, this need not be the case if ξ is not homogeneous.

The following lemma is a special case of [BP1, Proposition 2.2].

LEMMA 2.1. *Let A, B, and C be K-vector spaces and let $A \times B \to C$ be a bilinear map given by $a \times b \mapsto ab$. Assume that for all $b \in B \setminus \{0\}$ the annihilator $\operatorname{ann}_A(b)$ has infinite codimension in A. If B has finite dimension, then for any integer $N \geq 1$ there exists an N-dimensional subspace A' of A such that $A'B \cong A' \times B$.*

The following key technical lemma is a modified version of [W, Lemma 3.2].

LEMMA 2.2. *Suppose $\Delta_x(L) = 0$ and let V be a finite-dimensional homogeneous subspace of L, H a countable-dimensional subspace of L, and $N \geq 1$ a fixed integer. Then there exist*

(i) *a homogeneous basis $\mathscr{Y} = \{y_1, y_2, \ldots, y_q\}$ for V;*

(ii) *homogeneous elements $t_{ijk} \in L$ which are linearly independent modulo H, where $1 \leq i \leq q$, $1 \leq j \leq q$, $1 \leq k \leq N$;*

(iii) *homogeneous elements $x_{ik} \in L$, where $1 \leq i \leq q$, $1 \leq k \leq N$;*

(iv) *subsets S_i of $\{1, 2, \ldots, q\}$ containing i, for $1 \leq i \leq q$, with the properties that, for any i and k,*

(1) *if $j \in S_i$, then $[x_{ik}, y_j] = t_{ijk}$;*

(2) *if $j \notin S_i$, then $[x_{ik}, y_j]$ is in the span of those $t_{ij'k}$ with $j' \in S_i \setminus \{i\}$.*

Proof. For every $g \in G$, we define $W_g = \mathbb{D}_x(L_g) \cap V$ and note, by Lemma 1.2(i), that each W_g is homogeneous and that $\bigcap_{g \in G} W_g = 0$. By [W, Lemma 3.1], V has a homogeneous basis $\mathscr{Y} = \{y_1, y_2, \ldots, y_q\}$ such that for any $y \in \mathscr{Y}$, there exists some $g \in G$ with W_g contained in the K-subspace spanned by $\mathscr{Y} \setminus \{y\}$.

We show inductively for every m with $1 \leq m \leq q$, that the appropriate subsets S_i and homogeneous elements $x_{ik} \in L$ and $t_{ijk} \in L$ exist for $1 \leq i \leq m$, $1 \leq j \leq q$, $1 \leq k \leq N$ such that properties (1) and (2) are satisfied.

We begin with the $m = 1$ case. Here, if we let V_1 denote the span of $\mathscr{Y} \setminus \{y_1\}$, then there is some $g \in G$ such that $W_g \subseteq V_1$. Since W_g is finite-dimensional and homogeneous, Lemma 1.2(ii) implies that there exists a homogeneous subspace M_g of countable codimension in L_g with $[M_g, W_g] = 0$. Let R_1 be a subset of $\{2, \ldots, q\}$ such that $\{y_r \mid r \in R_1\}$ is a complementary basis for W_g in V_1. It is clear that if $S_1 = R_1 \cup \{1\}$, then $\{y_s \mid s \in S_1\}$ is a complementary basis for W_g in V and $1 \in S_1$.

Next let $A = M_g$, let B denote the span of $\{y_s \mid s \in S_1\}$, and let C be the quotient space L/H. Then the Lie bracket induces a bilinear map from $A \times B$ to C and we note that the annihilator of any nonzero $b \in B$ has infinite (in fact, uncountable) codimension in A. This is immediate since $W_g = \mathbb{D}_x(L_g) \cap V$, since $B \cap W_g = 0$, and since both $\dim_K L_g/M_g$ and $\dim_K H$ are countable. Therefore, we can apply Lemma 2.1 to conclude that there exist $x_{11}, \ldots, x_{1N} \in M_g$ such that, for $1 \leq k \leq N$ and $j \in S_1$, the elements $[x_{1k}, y_j] + H$ are linearly independent in L/H. In particular, if we let $t_{1jk} = [x_{1k}, y_j]$, for $1 \leq k \leq N$ and $j \in S_1$, then these elements are homogeneous and linearly independent modulo H. Finally, notice that if $j \notin S_1$, then $y_j \in V_1$ and V_1 is the K-linear span of W_g and those y_r with $r \in R_1$. Thus, since $[M_g, W_g] = 0$ and $x_{1k} \in M_g$, it follows that $[x_{1k}, y_j]$ is in the K-linear span of those $t_{1j'k}$ with $j' \in R_1 = S_1 \setminus \{1\}$.

We now suppose that $m > 1$ and that the appropriate S_i, x_{ik} and t_{ijk} exist for $1 \leq i \leq m - 1$. By slightly modifying the above argument, we prove that S_m, x_{mk}, and t_{mjk} also exist satisfying properties (1) and (2). To start with, let V_m denote the K-span of $\mathscr{Y} \setminus \{y_m\}$ and again observe that there exists some $h \in G$ with $W_h \subseteq V_m$. Furthermore, there exists a subspace M_h of countable codimension in L_h such that $[M_h, W_h] = 0$. As before, we let R_m be a subset of $\{1, \ldots, q\} \setminus \{m\}$ such that $\{y_r \mid r \in R_m\}$ is a complementary basis for W_h in V_m, and we set $S_m = R_m \cup \{m\}$.

Finally, let T_{m-1} be the subspace of L spanned by those t_{ijk} with $1 \leq i \leq m - 1$, and let $H' = H + T_{m-1}$. Then H' has countable dimension and, as above, the existence of the elements x_{mk} and t_{mjk} follows by applying Lemma 2.1 to the bilinear map $[\ ,\ ]$: $A \times B \to C$, where $A = M_h$, B is the K-linear span of the set $\{y_s \mid s \in S_m\}$, and $C = L/H'$. This completes the inductive step and the lemma is proved. ∎

The following is essentially [W, Lemma 3.3], which is the color version of [BP2, Lemma 3.4]. Note that the hypothesis implies that $a_i \leq N$ for each i, and hence all x_{ij} appearing in δ do indeed exist.

LEMMA 2.3. *Suppose that V is a q-dimensional homogeneous subspace of L and $N \geq 1$ is fixed. Let $\mathscr{Y} = \{y_i\}, \{x_{ik}\}$, $\mathscr{T} = \{t_{ijk}\}$, and $\{S_i\}$ be as in Lemma 2.2. Suppose that $b_1, b_2, \ldots, b_q$ are nonnegative integers with $N = b_1 + b_2 + \cdots + b_q$ and that $\xi = y_1^{b_1} y_2^{b_2} \cdots y_q^{b_q}$ is a straightened monomial in $U(L)$. Let*

$$\delta = x_{11} \cdots x_{1a_1} x_{21} \cdots x_{2a_2} \cdots x_{q1} \cdots x_{qa_q} \in U(L),$$

where the a_i are nonnegative integers with $N = a_1 + a_2 + \cdots + a_q$. Then

$$\xi^{\delta} = \xi_1 + \xi_2 + \xi_3,$$

where $\xi_1 \in \sum_{s=0}^{N-1} L^s \mathscr{Y} L^{N-s-1} = \mathscr{Y} L^{N-1} = L^{N-1} \mathscr{Y}$ and ξ_2 is a K-linear combination of straightened monomials of degree N in the elements of $\mathscr{T}$ having at least one factor t_{ijk} with $i \neq j$. Finally, if $a_i = b_i$ for all i, then

$$\xi_3 = \theta a_1! a_2! \cdots a_q! t_{111} \cdots t_{11a_1} t_{221} \cdots t_{22a_2} \cdots t_{qq1} \cdots t_{qqa_q}$$

for some $\theta \in K^*$, and $\xi_3 = 0$ otherwise.

With this, we can prove

PROPOSITION 2.4. *Assume that $\Delta_\infty(L) = 0$, let H be a countable-dimensional (restricted) homogeneous Lie color subalgebra of L, and let $0 \neq \alpha \in U(H)$ be homogeneous. Then there exist $\alpha', \alpha'' \in U(L)\alpha U(L)$*

such that α' *and* α'' *can be written based on* H *as*

$$\alpha' = \sum_{\nu} \alpha'_{\nu} \nu$$

$$\alpha'' = \sum_{\eta} \eta \alpha''_{\eta}$$

with some α'_{ν_0} *and* α''_{η_0} *equal to* 1.

Proof. Let V be a finite-dimensional homogeneous subspace of H containing the support of α with respect to some basis of H. If V has dimension q and if $N = \deg \alpha$, then there exists a homogeneous basis $\mathscr{Y} = \{y_1, y_2, \dots, y_q\}$ for V, homogeneous elements x_{ik} and t_{ijk}, and subsets S_i, for $1 \le i \le q$, $1 \le j \le q$, and $1 \le k \le N$, satisfying the conclusion of Lemma 2.2. We naturally order the elements of $\mathscr{Y}$ so that $y_1 < y_2 \cdots < y_q$ and extend $\mathscr{Y}$ to an ordered homogeneous basis $\mathscr{B}$ of H. With respect to this new basis, it is clear that α can be written as $\alpha = \sum_{\xi} c_{\xi} \xi$, where each ξ is a straightened monomial in the elements of $\mathscr{B}$, each $c_{\xi} \in K$, and where each ξ of degree $N = \deg \alpha$ involves only elements of $\mathscr{Y}$.

We recall that the elements t_{ijk} are linearly independent modulo H and that if $j \in S_i$ then $t_{ijk} = [x_{ik}, y_j]$. Thus, there exists a homogeneous complementary basis $\mathscr{T}$ for H in L containing all t_{ijk} and we may suppose that $\mathscr{T}$ is ordered in such a way that $t_{iik} < t_{i'i'k'}$ whenever $i < i'$, and $t_{iik} < t_{iik'}$ whenever $k < k'$.

Suppose $\kappa = y_1^{a_1} y_2^{a_2} \cdots y_q^{a_q}$ is a monomial of degree N appearing in the above expression for α, so $c_{\kappa} \ne 0$. Set $\delta = x_{11} \cdots x_{1a_1} x_{21} \cdots x_{2a_2} \cdots x_{q1} \cdots x_{qa_q}$ and write α^{δ} based on H using the complementary basis $\mathscr{T}$. We compute the (left) coefficient in $U(H)$ of the straightened monomial $\nu_0 = t_{111} \cdots t_{11a_1} t_{221} \cdots t_{22a_2} \cdots t_{qq1} \cdots t_{qqa_q}$. Since ν_0 has degree N and $\deg \alpha^{\delta} \le \deg \alpha = N$, it is clear that this coefficient must be contained in K.

Let ξ be a monomial which occurs in $\alpha = \sum_{\xi} c_{\xi} \xi$. If $\deg \xi < N$, then $\deg \xi^{\delta} \le \deg \xi < N$ and ξ^{δ} does not contribute to the coefficient of ν_0. Thus we can suppose that $\deg \xi = N$ and therefore that

$$\xi = y_1^{b_1} y_2^{b_2} \cdots y_q^{b_q}$$

for suitable nonnegative integers b_i with $b_1 + b_2 + \cdots + b_q = N = a_1 + a_2 + \cdots + a_q$. By Lemma 2.3, we can write

$$\xi^{\delta} = \xi_1 + \xi_2 + \xi_3$$

with the ξ_i's having appropriate properties. To start with,

$$\xi_1 \in \sum_{s=0}^{N-1} L^s \mathscr{Y} L^{N-s-1} = \mathscr{Y} L^{N-1} = L^{N-1}\mathscr{Y},$$

and thus if we write ξ_1 based on H, then all monomials in the complementary basis $\mathscr{T}$ will have degree at most $N-1$ since $\mathscr{Y} \subseteq H$. Hence ξ_1 does not contribute to the coefficient of ν_0. Next, ξ_2 is a K-linear combination of straightened monomials in the elements of $\mathscr{T}$, and none of these monomials is equal to ν_0. So again, ξ_2 does not contribute to the coefficient of ν_0. Of course, ξ_3 will not contribute either if $\xi_3 = 0$. In view of Lemma 2.3, this leaves only

$$\xi = \kappa = y_1^{a_1} y_2^{a_2} \cdots y_q^{a_q},$$

in which case

$$\xi_3 = \theta a_1! a_2! \cdots a_q! t_{111} \cdots t_{11a_1} t_{221} \cdots t_{22a_2} \cdots t_{qq1} \cdots t_{qqa_q}$$

for some $\theta \in K^*$. Thus the coefficient of ν_0 in α^δ is precisely equal to $c = c_\kappa \theta a_1! a_2! \cdots a_q!$, a nonzero element of K since L is restricted if char $K > 0$. Finally, since α is homogeneous,

$$\alpha' = c^{-1}\alpha^\delta \in U(L)\alpha U(L)$$

and this element has ν_0 coefficient equal to 1. Thus the first part is proved, and the existence of $\alpha'' \in U(L)\alpha U(L)$ follows in a similar manner. ∎

We can now obtain the main result of this section.

THEOREM 2.5. *If $\Delta_\infty(L) = 0$, then $U(L)$ is symmetrically closed.*

Proof. Since $\Delta_\infty(L) = 0$, it is certainly the case that $\Delta(L) = \Delta_L = 0$ and therefore, by [W, Theorem 5.3], $U(L)$ is prime. Thus $Q_s(U(L))$ exists and is an overring of $U(L)$. Now let $q \in Q_s(U(L))$ and choose a nonzero ideal I of $U(L)$ with $Iq, qI \subseteq U(L)$. Since $U(L)$ is prime, [CR, Corollary 3.4] implies that there exists a nonzero homogeneous ideal contained in I. Therefore we may fix some nonzero homogeneous element $\alpha \in I$, and we note that $\alpha U(L) q \subseteq Iq \subseteq U(L)$. The proof now proceeds in a series of two steps.

STEP 1. *There exists a countable-dimensional (restricted) homogeneous Lie color subalgebra H of L such that*

(i) $\alpha \in U(H)$,

(ii) $\mathrm{r}\cdot\mathrm{ann}_{U(H)}\alpha U(H) = 0$,

(iii) $\alpha U(H) q \subseteq U(H)$.

Proof. We first note that there exists a finitely generated and hence countable-dimensional (restricted) homogeneous Lie color subalgebra H_0 of L such that $\alpha \in U(H_0)$. We now construct an ascending sequence

$$H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$$

of countable-dimensional (restricted) homogeneous Lie color subalgebras of L such that

(a) $\mathrm{r} \cdot \mathrm{ann}_{U(H_n)} \alpha U(H_{n+1}) = 0$,

(b) $\alpha U(H_n) q \subseteq U(H_{n+1})$.

To this end, suppose H_n is given and note that $\alpha \in U(H_0) \subseteq U(H_n)$. By Proposition 2.4, $U(L)\alpha U(L)$ contains an element $\alpha'' = \sum_\eta \eta \alpha''_\eta$ written based on H_n with some $\alpha''_{\eta_0} = 1$. Of course, there exists a finite-dimensional homogeneous subspace S of L with $\alpha'' \in S^\infty \alpha S^\infty$.

Next, $\alpha U(H_n) q$ is a countable-dimensional subspace of $U(L)$, so there exists a countable-dimensional homogeneous subspace T of L with $\alpha U(H_n) q \subseteq T^\infty$. Now let H_{n+1} be the (restricted) homogeneous Lie color subalgebra of L generated by H_n, S, and T. Then H_{n+1} is countably generated and hence countable-dimensional. It remains to show that (a) and (b) are satisfied.

For (a), suppose $\beta \in U(H_n)$ with $\alpha U(H_{n+1})\beta = 0$. Then

$$U(H_{n+1})\alpha U(H_{n+1})\beta = 0$$

and, since $H_{n+1} \supseteq S$, we have

$$\alpha'' \in S^\infty \alpha S^\infty \subseteq U(H_{n+1})\alpha U(H_{n+1})$$

and $\alpha''\beta = 0$. But, if we write α'' based on H_n as above, then since $\beta \in U(H_n)$,

$$0 = \alpha''\beta = \sum_\eta \eta \alpha''_\eta \beta$$

implies that each $\alpha''_\eta \beta = 0$. In particular, since $\alpha''_{\eta_0} = 1$, we conclude that $\beta = 0$ and (a) is proved. For (b), we have $T \subseteq H_{n+1}$, so

$$\alpha U(H_n) q \subseteq T^\infty \subseteq U(H_{n+1}).$$

Thus the induction step is proved and the ascending sequence exists.

Finally, let $H = \bigcup_{n=0}^\infty H_n$. Then H is a countable-dimensional (restricted) homogeneous Lie color subalgebra of L and certainly $\alpha \in$

$U(H_0) \subseteq U(H)$. Furthermore if $\gamma \in U(H)$ with $\alpha U(H)\gamma = 0$, then $\gamma \in U(H_n)$ for some n, so $\alpha U(H_{n+1})\gamma = 0$ and condition (a) imply that $\gamma = 0$. Finally, $\alpha U(H)q \subseteq U(H)$ follows immediately from (b) since $U(H) = \bigcup_{n=0}^{\infty} U(H_n)$. ∎

STEP 2. $q \in U(L)$.

Proof. By Proposition 2.4, we can choose $\alpha' \in U(L)\alpha U(L) \subseteq I$ such that

$$\alpha' = \sum_{\nu} \alpha'_{\nu} \nu$$

can be written based on H with some $\alpha'_{\nu_0} = 1$. Now for any $\beta \in U(H)$,

$$(\alpha\beta q)\alpha' = \alpha\beta(q\alpha')$$

and we know that $\gamma = q\alpha' \in qI \subseteq U(L)$. In particular, if we write $\gamma = \sum_{\nu}\gamma_{\nu}\nu$ based on H, using the same complementary basis as for α', then

$$\sum_{\nu}(\alpha\beta q)\alpha'_{\nu}\nu = (\alpha\beta q)\alpha' = (\alpha\beta)\gamma = \sum_{\nu}(\alpha\beta)\gamma_{\nu}\nu.$$

But $\alpha\beta q \in U(H)$, so this yields

$$(\alpha\beta q)\alpha'_{\nu} = (\alpha\beta)\gamma_{\nu}$$

for all monomials ν. In particular, since $\alpha'_{\nu_0} = 1$, we have

$$\alpha\beta q = \alpha\beta\gamma_{\nu_0}$$

for all $\beta \in U(H)$. Thus $\alpha U(H)(q - \gamma_{\nu_0}) = 0$.

Finally, note that $\alpha U(H)$ is right regular in $U(H)$ and hence, by the uniqueness part of Lemma 1.1, $\alpha U(H)$ is right regular in $U(L)$. This then implies that $\alpha U(H)$ is right regular in $Q_s(U(L))$ and hence the equation $\alpha U(H)(q - \gamma_{\nu_0}) = 0$ yields $q - \gamma_{\nu_0} = 0$. Thus $q = \gamma_{\nu_0} \in U(H) \subseteq U(L)$, as required, and the theorem is proved. ∎

As we mentioned at the beginning of this section, the hypothesis $\Delta_{\chi}(L) = 0$ in Theorem 2.5 cannot be replaced by the weaker assumption $\Delta_L = 0$ because of restricted Lie algebra examples in characteristic $p > 0$ and Lie superalgebra examples in characteristic 0. However, we do not know whether $\Delta_L = 0$ implies that $U(L)$ is symmetrically closed when L is an ordinary Lie algebra and char $K = 0$.

3. COMPARING $\Delta(L)$ AND Δ_L

If L is a restricted Lie color algebra in characteristic $p > 0$, then it is not hard to show that $\Delta_L = \Delta(L)$. However, in characteristic 0, $\Delta(L)$ can be appreciably larger than Δ_L. Indeed, an example is given in [BP3] in which L is an infinite-dimensional Lie algebra, $\Delta(L)$ has codimension 1, and yet $\Delta_L = 0$. Recently, M. Wilson asked whether $[\Delta(L), \Delta(L)]$ is always contained in Δ_L. In this section, we give an affirmative answer to the question, and in fact we obtain a slight generalization. To this end, let $\mathbb{D}_L(\Delta(L)) = \{l \in L \mid \dim_K[\Delta(L), l] \text{ is finite}\}$. Then $\mathbb{D}_L(\Delta(L))$ is certainly a Lie color ideal of L containing $\Delta(L)$, and we prove

THEOREM 3.1. *If L is a Lie color algebra and $D = \mathbb{D}_L(\Delta(L))$, then $[\Delta(L), D] \subseteq \Delta_L$. In particular, $[\Delta(L), \Delta(L)] \subseteq \Delta_L$.*

Proof. By [W, Proposition 2.9], $\Delta_L = \bigcap_{l \in L_+} \Delta_l$, where

$$\Delta_l = \{v \in \Delta(L) \mid \mathrm{ad}_l \text{ is algebraic on } v\}.$$

Therefore, it suffices to show that $[\Delta(L), D] \subseteq \Delta_l$ for all $l \in L_+$. To this end, fix $0 \neq l \in L_g$, where $g \in G_+$, and let $x \in \Delta(L) \cap L_h$. Our goal is to show that ad_l is algebraic on $[x, D]$ and hence that $[x, D] \subseteq \Delta_l$. To start with, observe that $[x, D]$ is finite-dimensional since $x \in \Delta(L)$, and therefore

$$M = [x, D] + [x, D]^l + [x, D]^{l^2} + \cdots$$

is a finitely generated module for the polynomial ring $K[l]$, with l acting as ad_l. Thus, since $K[l]$ is a principal ideal domain, M is a finite direct sum of free $K[l]$-modules of rank 1 and the torsion module $M \cap \Delta_l$. We will show that $[x, D] \subseteq M$ has no nonzero projection to any of these free summands of rank 1. This will imply that $[x, D] \subseteq M \cap \Delta_l$ and, since $\Delta(L)$ is homogeneous, this will yield the result.

Suppose, by way of contradiction, that $M = Z \oplus W$, where Z is a free summand of rank 1, W is a complementary $K[l]$-module, and $[x, D] \nsubseteq W$. Let $\{z_0, z_1, z_2, \ldots\}$ be a K-basis for Z with $z_i^l = z_{i+1}$ and, for convenience, write

$$Z_m = Kz_0 + Kz_1 + \cdots + Kz_m$$

and $Z_{-1} = 0$. Since $[x, D] \subseteq M$ is finite-dimensional and $[x, D] \nsubseteq W$, there exists $n \geq 0$ minimal with the property that $[x, D] \subseteq Z_n \oplus W$. Furthermore, since $[x, D] \nsubseteq Z_{n-1} \oplus W$, we can clearly find an element $y \in D$ with

$$[x, y] - z_n \in Z_{n-1} \oplus W.$$

Write $x_0 = x$, set $x_i = x^{l^i} \in \Delta(L)$ for $i \geq 1$, and note that $x_i \in L_{g^i h}$.

We first prove by induction on i that $[x_i, D] \subseteq Z_{n+i} \oplus W$. Indeed, by the definition of n, this is certainly true for $i = 0$. Suppose now that this fact holds for i, and let $d \in D$. Then

$$[x_i, d] \in Z_{n+i} \oplus W,$$

and, by applying ad_l, we obtain

$$[x_{i+1}, d] + \epsilon(g, g^i h)[x_i, d^l] \in Z_{n+i+1} \oplus W.$$

But $[x_i, d^l] \in [x_i, D] \subseteq Z_{n+i} \oplus W$, so $[x_{i+1}, d] \in Z_{n+i+1} \oplus W$, and this claim is proved.

Next, we use an almost identical induction argument to show that

$$[x_i, y] - z_{n+i} \in Z_{n+i-1} \oplus W$$

for all $i \geq 0$. Indeed, by the choice of y, this certainly holds for $i = 0$. Now suppose that it holds for some $i \geq 0$. Then, by applying ad_l to the above displayed equation, we obtain

$$[x_{i+1}, y] + \epsilon(g, g^i h)[x_i, y^l] - z_{n+i+1} \in Z_{n+i} \oplus W.$$

But $[x_i, y^l] \in [x_i, D] \subseteq Z_{n+i} \oplus W$, so $[x_{i+1}, y] - z_{n+i+1} \in Z_{n+i} \oplus W$, and the second claim is proved.

Finally, since $[x_i, y] - z_{n+i} \in Z_{n+i-1} \oplus W$, it follows that the elements $[x_i, y]$ are linearly independent. Indeed, for any $j \geq 0$, we have

$$[x_0, y], [x_1, y], \ldots, [x_j, y] \in Z_{n+j} \oplus W,$$

while

$$[x_{j+1}, y] \notin Z_{n+j} \oplus W.$$

Thus $[\Delta(L), y]$ is infinite-dimensional, and this contradicts the fact that $y \in D = \mathbb{D}_L(\Delta(L))$. With this, the result follows. ∎

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