MULTIPLICATIVE JORDAN DECOMPOSITION IN GROUP RINGS OF 3-GROUPS, II

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Abstract. In this paper, we complete the classification of those finite 3-groups G whose integral group rings have the multiplicative Jordan decomposition property. If G is abelian, then it is clear that $\mathbb{Z}[G]$ satisfies MJD. In the nonabelian case, we show that $\mathbb{Z}[G]$ satisfies MJD if and only if G is one of the two nonabelian groups of order $3^3 = 27$.

1. Introduction

Let $\mathbb{Q}[G]$ denote the rational group algebra of the finite group G. Since $\mathbb{Q}$ is a perfect field, every element a of $\mathbb{Q}[G]$ has a unique additive Jordan decomposition $a = a_s + a_n$, where a_s is a semisimple element and where a_n commutes with a_s and is nilpotent. If a is a unit, then a_s is also invertible and $a = a_s(1 + a_s^{-1}a_n)$ is a product of a semisimple unit a_s and a commuting unipotent unit $a_u = 1 + a_s^{-1}a_n$. This is the unique multiplicative Jordan decomposition of a. Following [AHP] and [HPW], we say that $\mathbb{Z}[G]$ has the multiplicative Jordan decomposition property (MJD) if for every unit a of $\mathbb{Z}[G]$, its semisimple and unipotent parts are both contained in $\mathbb{Z}[G]$. For simplicity, we say that G satisfies MJD if its integral group ring $\mathbb{Z}[G]$ has that property.

If G is abelian or a Hamiltonian 2-group, then every element of $\mathbb{Q}[G]$ is semisimple. Thus every unit a of $\mathbb{Z}[G]$ is equal to its semisimple part and consequently $\mathbb{Z}[G]$ trivially satisfies MJD. In the non-Dedekind case, it appears that the MJD property is relatively rare. Indeed, the papers [AHP] and [HPW] showed that $\mathbb{Z}[G]$ and $\mathbb{Q}[G]$ must be quite restrictive. Furthermore, using numerous clever arguments, paper [HPW] was able to determine all nonabelian 2-groups that satisfy MJD. Specifically, these are the two nonabelian groups of order 8, five groups of order 16, four groups of order 32, and only the Hamiltonian groups of larger order.

Building on the work of [HPW], and using variants of many of the same arguments, [LP1] came close to determining all nonabelian 3-groups satisfying MJD. Specifically, these included the two nonabelian groups of order 3^3 and at most three groups of order $3^4 = 81$, namely

i. the central product of a cyclic group of order 9 with the nonabelian group of order 27 and period 3, or
ii. the group generated by x, y and z subject to the relations $x^9 = y^3 = 1$, $xy = yx$, $x^z = xy$, $y^z = yx^{-3}$ and $z^3 = x^3$, or
iii. the semidirect product $G = X \rtimes Y$, where X and Y are cyclic of order 9.

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In a later paper, [LP2], groups (ii) and (iii) were eliminated. Thus only the group (i) remained and the goal of Section 2 is to show that this group does not have MJD. With this, we obtain our main result, namely

Theorem 1.1. *Let G be a finite nonabelian 3-group. Then $\mathbb{Z}[G]$ has the MJD property, if and only if G has order $3^3 = 27$.*

2. The central product group of order 81

Let G_0 be the nonabelian 3-group of order 27 and period 3. Set $\mathfrak{Z}(G_0) = W = \langle w \rangle$, a cyclic group of order 3, and let $H = \langle h \rangle$ to be any nonnormal subgroup of G_0 of order 3. Then $N_0 = \mathfrak{N}_{G_0}(H) = W \times H$, and we choose two elements $x, y \in G_0 \setminus N_0$ so that x and y do not commute and with cosets $N_0x = N_0y$. Indeed, we can fix $x \in G_0 \setminus N_0$ with $h^x = wh$ and take y to be any element of $N_0x \setminus Wx$.

If T is a subgroup of G_0, then in the rational group algebra $\mathbb{Q}[G_0]$, we let $\widehat{T}$ denote the sum of the group elements of T. As is well known, $(1-t)\widehat{T} = 0$ for all $t \in T$, and $(\widehat{T})^2 = |T|\,\widehat{T}$. In particular, $(\widehat{H})^2 = 3\widehat{H}$ and $e = \widehat{H}/3$ is an idempotent in $\mathbb{Q}[G_0]$. Furthermore, set $s = (1-w)(x-y) \in \mathbb{Q}[G_0]$.

Lemma 2.1. *With the above notation, we have*

i. $ese = 0$.
ii. $s^2 = s^2e + ses + es^2$.
iii. $s^3 = 0$.

Proof. (i) If $t = x$ or y, then

$$(1-w)\widehat{H}t\widehat{H} = t(1-w)\widehat{H^t}\widehat{H} = t(1-w)\widehat{N_0} = 0$$

since $H^tH = N_0$ and $w \in N_0$. But $s = (1-w)(x-y)$ and $e = \widehat{H}/3$, so we conclude that $ese = 0$.

(ii) Multiplying by 3, it suffices to show that $3s^2 = s^2\widehat{H} + s\widehat{H}s + \widehat{H}s^2$. Now x and y are in the same coset of $N_0 = \mathfrak{N}_{G_0}(H)$, so by shifting the s terms to the left, it follows easily that

$$\begin{aligned} s^2\widehat{H} + s\widehat{H}s + \widehat{H}s^2 &= s^2(\widehat{H} + \widehat{H^x} + \widehat{H^{x^2}}) \\ &= s^2(\widehat{W} + \widehat{H} + \widehat{H^x} + \widehat{H^{x^2}}) \end{aligned}$$

since $(1-w)$ is a factor of s and $(1-w)\widehat{W} = 0$. But $\{W, H, H^x, H^{x^2}\}$ is a partition of N_0. In particular, in the sum of their hats, each nonidentity element of N_0 occurs precisely once, while the identity appears four times. Thus the sum of their hats is $3 + \widehat{N_0}$, and

$$s^2\widehat{H} + s\widehat{H}s + \widehat{H}s^2 = s^2(3 + \widehat{N_0}) = 3s^2$$

since $(1-w)$ is a factor of s, $w \in N_0$ and $(1-w)\widehat{N_0} = 0$.

(iii) Note that

$$\begin{aligned} (x-y)^3 &= x^3 - y^3 + (y^2x + yxy + xy^2) - (x^2y + xyx + yx^2) \\ &= (y^2x + yxy + xy^2) - (x^2y + xyx + yx^2) \end{aligned}$$

since $x^3 = y^3 = 1$. Furthermore, x and y do not commute and $W = G_0'$, so we have

$$y^2x + yxy + xy^2 = y^2x(1 + w + w^2) = y^2x\widehat{W}$$

and hence this term is annihilated by $1-w$. Similarly, $x^2y+xyx+yx^2$ is annihilated by $1-w$ and therefore $s^3=0$. □

We continue with the above notation. Furthermore, we let $G = Z * G_0$ be the central product of G_0 with the cyclic group $Z = \langle z\rangle$ of order 9. Here $\mathfrak{Z}(G_0) = W$ is identified with the subgroup of Z of order 3. In particular, we can assume that $w = z^3$. Now let b be any element of $\mathbb{Q}[Z]$ and define $u = u(b) \in \mathbb{Q}[G]$ by

$$u = 1 + (b-1)e + s.$$

Note that $N = \mathfrak{N}_G(H) = Z \times H$ and that $Nx = Ny$.

For convenience, let $\pi\colon \mathbb{Q}[G] \to \mathbb{Q}[G]$ be the projection to the elements with support in the coset $Zx^2 \subseteq G$. In other words, if $\alpha = \sum_{g\in G} a_g g \in \mathbb{Q}[G]$, then $\pi(\alpha) = \sum_{g\in Zx^2} a_g g$. It is clear that π is a $\mathbb{Q}[Z]$-bimodule map.

Lemma 2.2. *If b, u and π are as above, then*

i. $(u-1)^2(u-b) = 0$.
ii. $\pi((u-1)^2) = \pi((u-1)(u-b)) = \pi(s^2) = (1-w)^2x^2$.

Proof. Since b is central, all the factors in the displayed polynomial expressions commute. Furthermore, $u-1 = (b-1)e + s$ and $u-b = (b-1)(e-1)+s$. Of course, $e(e-1) = 0$.

(i) We have

$$\begin{aligned}(u-1)^2(u-b) &= [(b-1)e+s]^2[(b-1)(e-1)+s]\\ &= (b-1)^3e^2(e-1) + (b-1)^2[e^2s + (es+se)(e-1)]\\ &\qquad + (b-1)[(es+se)s + s^2(e-1)] + s^3\\ &= (b-1)^2[ese] + (b-1)[(es^2+ses+s^2e) - s^2] + s^3\\ &= 0\end{aligned}$$

since each of the coefficients of $(b-1)^i$ is zero by the preceding lemma.

(ii) Note that

$$(u-1)^2 = (b-1)^2e^2 + (b-1)(es+se) + s^2$$

and

$$(u-1)(u-b) = (b-1)^2e(e-1) + (b-1)[es + s(e-1)] + s^2.$$

Furthermore, the $(b-1)^2$ terms have support in the coset N, while the $(b-1)$ terms have support in the coset $Ny = Nx$, and these are different from Nx^2. Since $Zx^2 \subseteq Nx^2$, it follows that $\pi((u-1)^2) = \pi((u-1)(u-b)) = \pi(s^2)$.

Finally, since $s \in \mathbb{Q}[G_0]$ and $Z \cap G_0 = W$, it is clear that $\pi(s^2)$ is the projection of s^2 to the space of elements with support in the coset Wx^2. Now $G_0/W \cong C_3 \times C_3$ is generated by the images of x and y. Hence Wx^2, Wy^2 and $Wxy = Wyx$ are three distinct cosets of W. Since

$$s^2 = (1-w)^2(x^2 - xy - yx + y^2),$$

we conclude that $\pi(s^2) = (1-w)^2x^2$, as required. □

At this point it is appropriate to introduce additional assumptions on the central element $b \in \mathbb{Q}[Z]$ related to the integral group ring $\mathbb{Z}[Z]$.

Lemma 2.3. *Suppose $b \neq 1$ is a unit of $\mathbb{Z}[Z]$ with*

$$b \equiv 1 \bmod 3(1-w)\,\mathbb{Z}[Z].$$

Then $u = u(b)$ is a unit in $\mathbb{Z}[G]$ whose semisimple part is not in $\mathbb{Z}[G]$.

Proof. By the above displayed equation $b = 1 + 3(1-w)c$ for some $c \in \mathbb{Z}[Z]$ and hence, since $e = \widehat{H}/3$, we have

$$u = 1 + (b-1)e + s = 1 + (1-w)c\widehat{H} + s \in \mathbb{Z}[G].$$

Furthermore, by the preceding lemma, $(u-1)^2(u-b) = 0$ so

$$u[u^2 - (2+b)u + (1+2b)] = b.$$

In particular, since b is a unit in $\mathbb{Z}[Z]$,

$$u^{-1} = [u^2 - (2+b)u + (1+2b)]b^{-1} \in \mathbb{Z}[G]$$

and u is indeed a unit in $\mathbb{Z}[G]$.

To proceed further, we need to look a bit closer at the structure of $\mathbb{Q}[G]$. To start with, let $f = \widehat{W}/3$ be the central idempotent determined by W. Then

$$\mathbb{Q}[G] = f\mathbb{Q}[G] + (1-f)\mathbb{Q}[G]$$

is a ring direct sum with $f\mathbb{Q}[G] \cong \mathbb{Q}[G/W] = \mathbb{Q}[G/G']$. If we let $\theta' \colon \mathbb{Q}[G] \to f\mathbb{Q}[G]$ denote the projection to the first summand and if we identify $f\mathbb{Q}[G]$ with $\mathbb{Q}[G/W]$, then θ' is the natural epimorphism determined by the homomorphism $G \to G/W$. It follows that $\ker\theta' = (1-w)\mathbb{Q}[G]$. In particular, since

$$u = 1 + (1-w)c\widehat{H} + s = 1 + (1-w)[c\widehat{H} + (x-y)],$$

we see that $\theta'(u) = 1$. Similarly, $\theta'(b) = 1$.

On the other hand, the second summand $(1-f)\mathbb{Q}[G]$ is isomorphic to $\mathbf{M}_3(\mathbb{Q}[\varepsilon])$, the ring of 3×3 matrices over the cyclotomic field $\mathbb{Q}[\varepsilon]$, where ε is a primitive complex 9^{th} root of unity. Specifically, the homomorphism $\theta \colon \mathbb{Q}[G] \to (1-f)\mathbb{Q}[G] = \mathbf{M}_3(\mathbb{Q}[\varepsilon])$ can be described by

$$\theta(z) = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \qquad \theta(h) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad \theta(x) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where $\omega = \varepsilon^3$. Recall that x was chosen to satisfy $h^x = wh$. Note that $\theta(w) = \omega I$, $\theta(b) = \beta I$ for some $\beta \in \mathbb{Q}[\varepsilon]$ and $\theta(e) = e_{1,1}$, the matrix unit corresponding to the $(1,1)$-entry. It follows that $\theta(\mathbb{Z}[G]) \subseteq \mathbf{M}_3(\mathbb{Z}[\varepsilon])$. In particular, we have $\beta \in \mathbb{Z}[\varepsilon]$. Furthermore, since $\theta'(b) = 1$ and $b \neq 1$, it follows that $\theta(b) \neq I$ and hence $\beta \neq 1$.

Next, since $(u-1)^2(u-b) = 0$, by part (i) of the previous lemma, we have $(\theta(u) - I)^2(\theta(u) - \beta I) = 0$. Also, by that lemma, we have $(u-1)^2 \neq 0$ and $(u-1)(u-b) \neq 0$. But since $\theta'(u) = 1$, the latter two expressions map to 0 under θ'. Hence, they cannot map to 0 under θ and therefore we have $(\theta(u) - I)^2 \neq 0$ and $(\theta(u) - I)(\theta(u) - \beta I) \neq 0$. It follows that the minimal polynomial over $\mathbb{Q}[\varepsilon]$ of the 3×3 matrix $\theta(u) \in \mathbf{M}_3(\mathbb{Q}[\varepsilon])$ is precisely $(\zeta - 1)^2(\zeta - \beta)$, and therefore this must be the characteristic polynomial of $\theta(u)$. In particular, $\theta(u)$ has eigenvalue 1 with multiplicity two and eigenvalue $\beta \neq 1$ with multiplicity one.

Now let u_s be the semisimple part of the unit u in $\mathbb{Q}[G]$. Then $\theta(u_s)$ is the semisimple part of $\theta(u) \in \mathbf{M}_3(\mathbb{Q}[\varepsilon])$. Note that $\theta(u)$ is similar to

$$\begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

for some element $* \in \mathbb{Q}[\varepsilon]$ and that, under this same similarity, $\theta(u_s)$ becomes the diagonal matrix $\operatorname{diag}(1, 1, \beta)$. In particular, we see that $(\beta - 1)(\theta(u_s) - I) = (\theta(u) - I)^2$. In other words, $(b-1)(u_s - 1)$ and $(u-1)^2$ have the same image under θ. But they also have the same image under θ' since both expressions map to 0. Thus we see that

$$(b - 1)(u_s - 1) = (u - 1)^2.$$

Suppose, by way of contradiction, that $u_s - 1 \in \mathbb{Z}[G]$ and write $\pi(u_s - 1) = dx^2$ for some $d \in \mathbb{Z}[Z]$. Then applying the projection π to the above displayed equation, we obtain from Lemma 2.2(ii)

$$(b - 1)dx^2 = \pi((b - 1)(u_s - 1)) = \pi((u - 1)^2) = (1 - w)^2 x^2.$$

Thus $(b - 1)d = (1 - w)^2$, an equation in $\mathbb{Z}[Z]$.

Finally, recall that $b - 1 = 3(1 - w)c$ for some $c \in \mathbb{Z}[Z]$. Then the equation in $\mathbb{Z}[Z]$ becomes $3(1 - w)cd = (1 - w)^2$. Applying the homomorphism θ, which we can now view as mapping $\mathbb{Z}[Z]$ to $\mathbb{Z}[\varepsilon]$, we obtain

$$3(1 - \omega)\theta(cd) = \theta((1 - w)^2) = (1 - \omega)^2 = -3\omega$$

since $1 + \omega + \omega^2 = 0$. Canceling the factor of 3, we see that $(1 - \omega)\theta(cd) = -\omega$ and hence $1 - \omega$ is a unit in $\mathbb{Z}[\varepsilon]$, contradiction. Indeed, if $1 - \omega$ is invertible, then so is its complex conjugate $1 - \bar{\omega}$, and hence so is $(1 - \omega)(1 - \bar{\omega}) = 3$, which is surely not the case. Thus $u_s \notin \mathbb{Z}[G]$. □

It is now a simple matter to prove

Theorem 2.4. *The group $G = Z * G_0$ of order 81 does not have MJD.*

Proof. In view of the preceding lemma, it suffices to find a unit $1 \neq b \in \mathbb{Z}[Z]$ satisfying $b \equiv 1 \bmod 3(1 - w)\,\mathbb{Z}[Z]$. To this end, by [AP], we know that $\mathbb{Z}[Z]$ has a unit b_1 of infinite multiplicative order. Since the units in $\mathbb{Z}[Z/W] = \mathbb{Z}[C_3]$ are all trivial by [H], that is $\pm$ a group element, it follows that we can multiply b_1 by $\pm$ a group element of Z to guarantee that the image of b_1 in $\mathbb{Z}[Z/W]$ is equal to 1. Note that, in this process, b_1 still has infinite multiplicative order and $b_1 \equiv 1 \bmod (1 - w)\,\mathbb{Z}[Z]$. Thus $b_1 = 1 + (1 - w)c_1$ for some suitable $c_1 \in \mathbb{Z}[Z]$.

Next, since

$$(1 - w)^3 = 1 - 3w + 3w^2 - w^3 = -3w(1 - w)$$

and since $\mathbb{Z}[Z]$ is commutative, it follows that

$$b_1^3 = (1 + (1 - w)c_1)^3 \equiv 1 \bmod 3(1 - w)\,\mathbb{Z}[Z].$$

Furthermore, b_1^3 is a unit not equal to 1 since, by assumption, the unit b_1 has infinite multiplicative order. Thus we can take $b = b_1^3$. □

As a concrete example, following [AP], we can take

$$b_1 = 1 - 2(z + z^8) + (z^2 + z^7) + (z^4 + z^5)$$

with inverse

$$b_1^{-1} = -5 - (z + z^8) + 5(z^2 + z^7) + 3(z^3 + z^6) - 4(z^4 + z^5).$$

Then b_1 has infinite multiplicative order since, by [H], the units of finite order in the integral group ring of any abelian group are trivial, that is $\pm$ group elements. Furthermore, it is easy to verify that $b_1 \equiv 1 \bmod (1-w)\,\mathbb{Z}[Z]$. Indeed, $b_1 = 1 + (1-w)c_1$ with

$$c_1 = -2z + z^2 - z^4 + 2z^5.$$

Thus, by the above, we can take

$$b = b_1^3 = 55 - 69(z + z^8) + 48(z^2 + z^7) - 27(z^3 + z^6) + 21(z^4 + z^5).$$

Then $b = 1 + 3(1-w)c$, where

$$c = 18 - 23(z - z^5) + 16(z^2 - z^4) + 9z^3$$

and therefore

$$\begin{aligned} u = u(b) &= 1 + (1-w)c\widehat{H} + s \\ &= 1 + (1-w)[(18 - 23(z - z^5) + 16(z^2 - z^4) + 9z^3)\widehat{H} + (x - y)] \end{aligned}$$

yields an appropriate counterexample to the MJD property.

In view of our comments in the Introduction, Theorem 1.1 now follows immediately from Theorem 2.4 and the appropriate earlier results in [LP1] and [LP2].

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