

## INVARIANT IDEALS OF ABELIAN GROUP ALGEBRAS AND REPRESENTATIONS OF GROUPS OF LIE TYPE

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*Dedicated to the memory of our friend, Richard E. Phillips*

ABSTRACT. This paper contributes to the general study of ideal lattices in group algebras of infinite groups. In recent years, the second author has extensively studied this problem for  $G$  an infinite locally finite simple group. It now appears that the next stage in the general problem is the case of abelian-by-simple groups. Some basic results reduce this problem to that of characterizing the ideals of abelian group algebras stable under certain (simple) automorphism groups. Here we begin the analysis in the case where the abelian group  $A$  is the additive group of a finite-dimensional vector space  $V$  over a locally finite field  $F$  of prime characteristic  $p$ , and the automorphism group  $G$  is a simple infinite absolutely irreducible subgroup of  $GL(V)$ . Thus  $G$  is isomorphic to an infinite simple periodic group of Lie type, and  $G$  is realized in  $GL(V)$  via a twisted tensor product  $\phi$  of infinitesimally irreducible representations. If  $S$  is a Sylow  $p$ -subgroup of  $G$  and if  $\langle v \rangle$  is the unique line in  $V$  stabilized by  $S$ , then the approach here requires a precise understanding of the linear character associated with the action of a maximal torus  $T_G$  on  $\langle v \rangle$ . At present, we are able to handle the case where  $\phi$  is a rational representation with character field equal to  $F$ .

### 1. INTRODUCTION

This paper contributes to the general study of ideal lattices in group algebras of infinite groups. Some machinery for this was developed in the 70's to handle the group algebras of nilpotent and solvable groups, and properties of the Jacobson radical. Most of this is reflected in the book [8]. More recent progress is based on the idea of using asymptotic properties of representations of finite groups to analyze the group algebras of locally finite groups, see [13]. Results based on this idea mostly relate to the group algebras of simple locally finite groups. Although the solution for simple groups is still incomplete, the next stage in the general problem is probably the case of abelian-by-simple groups. Some basic results reduce this problem to that of characterizing the ideals of abelian group algebras stable under certain (simple) automorphism groups. Although one expects such a problem to be easier than the study of simple group algebras, our experience does not promise a quick answer. In this paper, we start the analysis in the (possibly) simplest case

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where the abelian group  $A$  is the additive group of a vector space  $V$  over a field of prime characteristic and the automorphism group  $G$  is a simple infinite absolutely irreducible subgroup of  $GL(V)$ .

Such groups  $G$  are known in some sense. Indeed,  $G$  is isomorphic to an infinite simple periodic group of Lie type and  $G$  is realized in  $GL(V)$  via a representation  $\phi$  that is a twisted tensor product of infinitesimally irreducible representations of  $G$  (i.e., those which arise from representations of the associated Lie algebra, see Section 3 for details). In spite of the fact that this data is well-understood, we have only been able to handle the case where  $\phi$  is rational (in the sense of algebraic group theory).

In our main result, stated below, we assume that  $V$  is a vector space over the field generated by the traces of all elements  $g \in G \subseteq GL(V)$ . This is equivalent to  $V$  being irreducible as an  $\mathbf{F}_p G$ -module where  $\mathbf{F}_p$  is the field of  $p$  elements. Such an assumption is exactly what one naturally wants for the study of the problem in general when  $G$  is not necessarily linear.

**Theorem 1.1.** *Let  $G$  be an infinite locally finite quasi-simple group of Lie type and let  $\phi : G \rightarrow GL(n, F)$  be a rational irreducible representation. Suppose that  $F$  is generated by the values  $\chi(g)$  for  $g \in G$ , where  $\chi$  is the character of  $\phi$ . Let  $V$  be the  $FG$ -module associated with  $\phi$ , and let  $A = V^+$  be the additive group of  $V$ . If  $K$  is a field of characteristic different from that of  $F$ , then  $KA$  and  $\text{Aug}(KA)$  are the only non-zero  $G$ -stable ideals of the group algebra  $KA$ .*

We remark that the recent paper of Brookes and Evans [3] considers the analogous problem when  $G = GL(V)$  and  $F$  is an arbitrary field (see also [7, Example 3.9]). Some of our results are also true under more general settings. Indeed, suppose  $F$  is arbitrary,  $G$  is an algebraic group over  $F$ , and that the highest weight vector  $v$  of  $\phi$  satisfies  $G_v \cdot v = Fv$ , where  $G_v$  is the stabilizer in  $G$  of the line  $\langle v \rangle$ . Then our argument reduces the problem to the case considered in [3] with  $\dim V = 1$ .

Most notation we require is introduced in Section 2. We note that all groups considered here are finite or locally finite, and that all representations are finite dimensional.

## 2. PRELIMINARIES ABOUT REPRESENTATIONS OF GROUPS OF LIE TYPE

In this section we refine some general results on representations of Chevalley groups. A quasi-simple Chevalley group  $G$  is determined by a Dynkin diagram and a pair  $P, \theta$  where  $P$  is a field (finite or locally finite in our situation) and  $\theta$  a Galois automorphism of  $P$ . We shall call  $P$  the *defining field* of  $G$ . If  $\theta = 1$ , then  $G$  is called normal or untwisted. In addition, we have  $\theta^2 = 1$  for the twisted groups  ${}^2A_r$ ,  ${}^2D_r$ ,  ${}^2E_6$ , and  $\theta^3 = 1$  for  ${}^3D_4$ . We call these the twisted groups of the first kind. For the twisted groups  ${}^2B_2$ ,  ${}^2F_4$ , and  ${}^2G_2$ , the automorphism  $\theta$  satisfies the equality  $\theta^2(x)x^p = 1$  for all  $x \in P$ , where  $p = \text{char } P = 2$  for  ${}^2B_2$  and  ${}^2F_4$ , and  $p = \text{char } P = 3$  for  ${}^2G_2$ . We call these the twisted groups of the second kind. Observe that the condition  $\theta^2(x)x^p = 1$  implies that  $\theta$  is of infinite order when  $P$  is infinite.

Throughout this paper,  $\mathbf{F}_p$  stands for the field of  $p$  elements and  $\bar{\mathbf{F}}_p$  for its algebraic closure.

Let  $G$  be given with  $p = \text{char } P$  and let  $\bar{G}$  be a simple, simply connected Chevalley group over  $\bar{\mathbf{F}}_p$  whose Dynkin diagram is that of  $G$ . We denote by  $r$  the rank of

$\bar{G}$  and by  $\text{Irr}(\bar{G})$  the set of all irreducible rational representations of  $\bar{G}$  (up to equivalence). Then  $\text{Irr}(\bar{G})$  is parameterized by highest weights which are simply strings  $(a_1, \dots, a_r)$  of non-negative integers. A highest weight is traditionally recorded as a linear combination  $a_1\omega_1 + \dots + a_r\omega_r$ , where  $\{\omega_1, \dots, \omega_r\}$  is a particular basis of the vector space  $\Omega = \Omega(\bar{G})$ , called the weight space. If  $p > 0$ , there is a further refinement of this parameterization. Indeed, each coefficient  $a_i$  can be expressed as  $a_i = b_1 + pb_2 + \dots + p^{n-1}b_n$  for some  $n = n(a_i)$ , where  $0 \leq b_j < p$  for  $j = 1, \dots, n$ , and  $b_n \neq 0$  unless  $a_i = 0$  in which case we set  $n(a_i) = n(0) = 0$ . This presentation is unique. The weights  $(a_1, \dots, a_r)$  with  $0 \leq a_i < p$  are called  $p$ -weights. Therefore, each weight  $\lambda = (a_1, \dots, a_r)$  can be uniquely expressed as  $\lambda_1 + \lambda_2 p + \dots + \lambda_n p^{n-1}$  where  $\lambda_1, \dots, \lambda_n$  are  $p$ -weights and  $n$  is now  $\max_i n(a_i)$ . Let  $f$  denote the Frobenius automorphism of  $\bar{\mathbf{F}}_p$  defined by  $x \mapsto x^p$  for  $x \in \bar{\mathbf{F}}_p$ . Clearly,  $f$  can be extended to the matrix ring  $M(k, \bar{\mathbf{F}}_p)$  for each  $k$ , and it is known that the group  $\bar{G}$  can be realized as a matrix group over  $\bar{\mathbf{F}}_p$ , via a rational representation  $\mu$  defined over  $\mathbf{F}_p$ , such that  $f(\bar{G}) = \bar{G}$ . The action of  $f$  on  $\bar{G}$  so defined does not depend upon the choice of  $\mu$ , so  $f$  becomes a well-defined automorphism of  $\bar{G}$ . For  $\phi \in \text{Irr}(\bar{G})$ , denote by  $\phi^f$  the  $f$ -twist of  $\phi$ , i.e.,  $\phi^f(g) = \phi(f(g))$ . Then the highest weight of  $\phi^f$  is equal to  $\lambda_1 p + \lambda_2 p^2 + \dots + \lambda_n p^n$  when  $\lambda_1 + \lambda_2 p + \dots + \lambda_n p^{n-1}$  is the highest weight of  $\phi$ .

Let  $q = p^t$ . Then the weights  $a_1\omega_1 + \dots + a_r\omega_r$  such that  $0 \leq a_i < q$  are called  $q$ -weights, and these all have shape  $\lambda_1 + \lambda_2 p + \dots + \lambda_t p^{t-1}$  where  $\lambda_1, \dots, \lambda_t$  are  $p$ -weights. Let  $\text{Irr}_q(\bar{G})$  denote the set of irreducible representations of  $\bar{G}$  whose highest weights are  $q$ -weights. If  $G = G(q)$  is either a non-twisted Chevalley group or one of the groups  ${}^2A_r(q)$ ,  ${}^2D_r(q)$ ,  ${}^2E_6(q)$ ,  ${}^3D_4(q)$ , then the set  $\phi|G$ , where  $\phi$  runs through the elements of  $\text{Irr}_q(\bar{G})$  is exactly the full set of pairwise non-equivalent irreducible  $\bar{\mathbf{F}}_p$ -representations of  $G$ . In particular, they are parameterized by  $q$ -weights. Suppose  $\rho$  is an irreducible representation of  $G$  and let  $\lambda$  be its highest weight. Then the field automorphism  $f$  twists  $\rho$ , and the  $q$ -weight of  $\rho^f$  is  $p\lambda$  modulo  $q$  when  $G$  is of normal type. This means that if  $\lambda = \lambda_1 + \lambda_2 p + \dots + \lambda_t p^{t-1}$ , then  $\lambda^f = \lambda_t + \lambda_1 p + \lambda_2 p^2 + \dots + \lambda_{t-1} p^{t-1}$ . On the other hand, if  $G$  is twisted, then  $\lambda^f = \tau(\lambda_t) + \lambda_1 p + \lambda_2 p^2 + \dots + \lambda_{t-1} p^{t-1}$  where  $\tau$  is a permutation of the set  $\{1, \dots, r\}$  such that  $|\tau| = |\theta|$ , and  $\tau$  acts on weights via  $\tau(a_1, \dots, a_r) = (a_{\tau(1)}, \dots, a_{\tau(r)})$ . For  ${}^2A_r(q)$  we have  $\tau(i) = r + 1 - i$ , for  ${}^2D_r(q)$  we have  $\tau$  equal to the transposition interchanging  $r - 1$  and  $r$ , for  ${}^2E_6(q)$  we have  $\tau(1, 2, 3, 4, 5, 6) = (6, 2, 5, 4, 3, 1)$ , and when  ${}^3D_4(q)$  we have  $\tau(1, 2, 3, 4) = (4, 2, 1, 3)$ . For groups of type  ${}^2B_2$ ,  ${}^2F_4$ , and  ${}^2G_2$ , the theory is slightly more complicated. However, in general, each irreducible representation of  $G$  extends to a representation of  $\bar{G}$ , see [10, Theorem 43].

Let  $S_p$  denote a Sylow  $p$ -subgroup of  $G$ . For finite  $G$  (as well as for infinite groups like  $\bar{G}$ ), the normalizer  $N_G(S_p)$  splits (quasi-splits, to be more precise) as  $N_G(S_p) = T_G \cdot S_p$ , where  $T_G$  is a subgroup called a maximal torus. Moreover, if  $f(G) = G$ , then  $S_p$  and  $T_G$  can both be chosen to be  $f$ -stable.

Let  $V$  be the  $\bar{\mathbf{F}}_p G$ -module associated with an irreducible representation  $\phi$  of  $G$ . By the above,  $\phi$  extends to a representation of  $\bar{G}$  for which we keep the notation  $\phi$  as well. By [5, Theorem 4.3] there is a unique line  $\langle v \rangle$  in  $V$  stabilized by  $S_p$ . This line is also stable under the Sylow  $p$ -subgroup of  $\bar{G}$  containing  $S_p$ , and  $v$  is a vector of highest weight for  $\phi(\bar{G})$ . In particular, if  $G_v$  denotes the stabilizer of the line  $\langle v \rangle$  in  $G$ , then there is a unique line  $\langle v \rangle$  in  $V$  with  $S_p \subseteq G_v$ . The map  $v \mapsto gv$  for

$g \in G_v$  is a 1-dimensional representation of  $G_v$  whose character we denote by  $\eta$ . For the following, see [5, Theorem 6.15].

**Lemma 2.1.** *Let  $V$  and  $V'$  be two irreducible  $\bar{\mathbb{F}}_p G$ -modules, and let  $v \in V$ ,  $v' \in V'$  be non-zero vectors fixed by the Sylow  $p$ -subgroup  $S_p$ . If  $\eta$  and  $\eta'$  are the respective characters of  $G_v$  and  $G_{v'}$ , then  $V \cong V'$  if and only if  $G_v = G_{v'}$  and  $\eta = \eta'$ .*

An important fact which is also a consequence of the general theory is that  $G_v = T_G \cdot C_G(v)$ , so  $\eta$  is essentially a character of  $T_G$ . We need to compute  $\eta$  in terms of  $\phi$  explicitly. Let  $\bar{T}$  denote a maximal torus of  $\bar{G}$  containing  $T_G$ .

Let  $\alpha_1, \dots, \alpha_r$  be simple roots of  $\bar{G}$  with  $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$ , see [10, §3]. Let  $h_\alpha(\bar{\mathbb{F}}_p^*)$  denote the 1-dimensional subtorus of  $\bar{T}$  corresponding to a root  $\alpha$  and, for brevity, set  $h_i = h_{\alpha_i}$ . Then each element of  $t \in \bar{T}$  can be (uniquely) expressed as  $\prod_{i=1}^r h_i(t_i)$  for some  $t_i \in \bar{\mathbb{F}}_p^*$ . As  $v$  is also a weight vector for  $\bar{G}$ , let  $\lambda = \sum_i a_i \omega_i$  be the weight of  $v$ . Then  $tv = \prod_{i=1}^r t_i^{a_i} v$ , see [10, Lemma 19], and therefore, to determine  $\eta(T_G)v$  one has to refine the expressions for the elements of  $T_G$  in terms of  $h_i(t_i)$ . This description is given in [4, §13.7].

If  $G$  is untwisted, then  $\eta(t) = \prod_{i=1}^r t_i^{a_i}$  where  $t_i$  are arbitrary elements in  $P^*$ . The situation for twisted groups is more delicate. We begin with the groups of first kind  ${}^2A_r(P)$ ,  ${}^2D_r(P)$ ,  ${}^2E_6(P)$ , and  ${}^3D_4(P)$ . Denote by  $P_\theta$  the subfield of  $\theta$ -fixed elements in  $P$ .

For  ${}^3D_4(P)$ , the element  $\prod_{i=1}^4 h_i(t_i)$  belongs to  $T_G$  if and only if  $t_2 \in P_\theta$  and  $t_1 \in P$ ,  $t_3 = \theta(t_1)$ ,  $t_4 = \theta^2(t_1)$ . For the unitary group  ${}^2A_{2r} \cong SU(2r+1, P)$ , the element  $t$  has shape  $\prod_{i=1}^r h_i(t_i) h_{2r+1-i}(\theta(t_i))$ , and for  ${}^2A_{2r-1} \cong SU(2r, P)$ , we have  $t = h_r(t_r) \cdot \prod_{i=1}^{r-1} h_i(t_i) h_{2r-i}(\theta(t_i))$ . Here the  $t_i$  run through all elements of  $P$ , except that  $t_r$  runs over  $P_\theta$  if  $G = {}^2A_{2r-1}$ .

For  $G = {}^2D_r(P)$ , we have  $t = h_{r-1}(\theta(t_r)) h_r(t_r) \prod_{i=1}^{r-1} h_i(t_i)$  where  $t_r \in P$  and  $t_i \in P_\theta$  for  $i = 1, \dots, r-1$ .

For  $G = {}^2E_6(P)$ , we have (using the ordering of simple roots as in [1])  $t = h_{\alpha_1}(t_1) h_{\alpha_6}(\theta(t_1)) h_{\alpha_3}(t_3) h_{\alpha_5}(\theta(t_3)) h_{\alpha_2}(t_2) h_{\alpha_4}(t_4)$  where  $t_1, t_3$  run over  $P$  and  $t_2, t_4$  run over  $P_\theta$ .

Next, we turn to the groups of type  ${}^2B_2(P)$ ,  ${}^2F_4(P)$ , and  ${}^2G_2(P)$ . They are subgroups of  $B_2(P)$ ,  $F_4(P)$ , and  $G_2(P)$ , respectively, where  $\text{char}(P) = 2$  in the first two cases and  $\text{char}(P) = 3$  in the third. The ordering of simple roots is chosen so that  $|\alpha_1| > |\alpha_2|$  for the  $B_2$  case,  $|\alpha_1| < |\alpha_2|$  for  $G_2$ , and  $|\alpha_1| = |\alpha_2| > |\alpha_3| = |\alpha_4|$  for  $F_4$ . According to [4, §13.7.4],  $t = h_{\alpha_1}(\theta(t_2^2)) h_{\alpha_2}(t_2)$  for  $G = {}^2B_2$ , and  $t = h_{\alpha_1}(t_1) h_{\alpha_2}(\theta(t_1^3))$  for  $G = {}^2G_2$  where  $t_1, t_2 \in P$ . If  $G = {}^2F_4(P)$ , then  $t = h_{\alpha_1}(\theta(t_4^2)) h_{\alpha_4}(t_4) h_{\alpha_2}(\theta(t_3^2)) h_{\alpha_3}(t_3)$  where  $t_3, t_4 \in P$ .

Thus, if  $v$  is a vector of weight  $\lambda = \sum_i a_i \omega_i$  and  $t \in T_G$  is expressed as above, then we obtain the following formulae.

**Lemma 2.2.** *With the above notation, the character  $\eta$  of  $T_G$  has the following shape:*

$$\begin{aligned} \eta(t) &= \prod_{i=1}^r t_i^{a_i} && \text{if } G \text{ is untwisted;} \\ \eta(t) &= \prod_{i=1}^r t_i^{a_i} \theta(t_i^{a_{2r+1-i}}) && \text{if } G = {}^2A_{2r}(P) \cong SU(2r+1, P); \\ \eta(t) &= t_r^{a_r} \cdot \prod_{i=1}^{r-1} t_i^{a_i} \theta(t_i^{a_{2r-i}}) && \text{if } G = {}^2A_{2r-1}(P) \cong SU(2r, P); \\ \eta(t) &= t_r^{a_r-1} \theta(t_r^{a_r}) \prod_{i=1}^{r-1} t_i^{a_i} && \text{if } G = {}^2D_r(P) \cong \Omega^-(2r, P); \\ \eta(t) &= t_1^{a_1} \theta(t_1^{a_6}) t_3^{a_3} \theta(t_3^{a_5}) t_2^{a_2} t_4^{a_4} && \text{if } G = {}^2E_6(P); \end{aligned}$$

$$\begin{aligned}
\eta(t) &= t_1^{a_1} \theta(t_1^{a_3}) \theta^2(t_1^{a_4}) t_2^{a_2} && \text{if } G = {}^3D_4(P); \\
\eta(t) &= t_1^{a_1} \theta(t_1^{3a_2}) && \text{if } G = {}^2G_2(P); \\
\eta(t) &= \theta(t_2^{2a_1}) t_2^{a_2} && \text{if } G = {}^2B_2(P); \\
\eta(t) &= \theta(t_4^{2a_1}) t_4^{a_4} \theta(t_3^{2a_2}) t_3^{a_3} && \text{if } G = {}^2F_4(P).
\end{aligned}$$

Remark. The values  $a_1, \dots, a_r$  are not always arbitrary non-zero integers. However this is not essential for our discussion.

**Lemma 2.3.** *Let  $P$  be a field of characteristic  $p > 0$  and let  $V$  be a vector space of finite dimension  $n$  over  $P$ . If  $G \subseteq GL(V) = GL(n, P)$  is an absolutely irreducible locally finite linear group, and if  $F$  is the subfield of  $P$  generated by all the traces of elements of  $G$ , then  $G$  is conjugate in  $GL(n, P)$  to a subgroup of  $GL(n, F)$ .*

Proof. This is well-known, see [12] or [14].

**Lemma 2.4.** *Let  $G$  be a locally cyclic group. If  $m \neq 0$  is an integer, then the index  $|G : G^m|$  is finite.*

Proof. The quotient group  $G/G^m$  is of exponent dividing  $m$  and it is also locally cyclic. Hence it is cyclic of order dividing  $m$ .

**Lemma 2.5.** *Let  $X, L$  be normal subgroups of a group  $G$ . If  $|G : XL| < \infty$  and  $|L : (X \cap L)| < \infty$ , then  $|G : X| < \infty$ .*

This is obvious since  $XL/X \cong L/(X \cap L)$ .

**Lemma 2.6.** *Let  $G$  be a locally cyclic group and let  $\theta$  be an automorphism of  $G$  of order 2. For  $i, j \in \mathbf{Z}$ , set  $X = \{\theta(g^i)g^j\}_{g \in G}$ . If  $i \pm j \neq 0$ , then  $|G : X| < \infty$ .*

Proof. Assume that  $i \pm j \neq 0$  and set  $L = \{\theta(g^i)g^i\}_{g \in G}$ . Since  $\theta^2 = 1$ , we have  $\theta(l) = l$  for all  $l \in L$ , and therefore  $\theta(l^i)l^j = l^{i+j} \in X \cap L$ . Note that, by Lemma 2.4, the subgroup  $\{l^{i+j}\}_{l \in L}$  is of finite index in  $L$ , since  $i + j \neq 0$  and  $L$  is locally cyclic. Therefore,  $L/(X \cap L)$  is finite. Furthermore,  $\theta(g^i)g^i \cdot g^{j-i} = \theta(g^i)g^j \in X$  for all  $g \in G$ . Hence  $g^{j-i} \in XL$  and, since  $G$  is locally cyclic and  $j - i \neq 0$ , we have  $|G : XL| < \infty$ . By Lemma 2.5, we conclude that  $|G : X| < \infty$ .

**Lemma 2.7.** *Let  $G$  be a locally cyclic group, let  $\theta$  be an automorphism of  $G$  of order 3, and let  $i, j, k \in \mathbf{Z}$ .*

(1) *If  $K_{j,k} = \{\theta(g^j)g^k\}_{g \in G}$ , and  $j + k \neq 0$ , then  $|G : K_{j,k}| < \infty$ .*

(2) *If  $X = \{\theta^2(g^i)\theta(g^j)g^k\}_{g \in G}$ , then  $|G : X| < \infty$  unless we have  $i = j = k$  or  $i + j + k = 0$ .*

Proof. (1) Set  $K = K_{j,k}$ . For all  $g \in G$ , we have  $\theta(g^j)g^k \in K$ , so  $\theta(g^{jk})g^{k^2} \in K_{j,k}$ . Moreover,  $\theta^2(g^k)g^j = \theta(\theta^2(g^j)\theta^2(g^k)) \in K$ , so  $\theta^2(g^{jk})g^{j^2} \in K$ . Multiplying these two expressions, we get  $\theta^2(g^{jk})\theta(g^{jk})g^{jk}g^{k^2+j^2-jk} \in K$ . In particular, if we set  $L_{jk} = \{\theta^2(g^{jk})\theta(g^{jk})g^{jk}\}_{g \in G}$ , then  $g^{k^2+j^2-jk} \in KL_{jk}$  for each  $g \in G$ . Observe that  $k^2+j^2-jk \neq 0$ , since otherwise  $k = j = 0$ . Hence, by Lemma 2.4,  $|G : KL_{jk}| < \infty$ . Now clearly,  $\theta(l) = l$  for  $l \in L_{jk}$ , and therefore  $\theta(l^j)l^k = l^{j+k} \in K \cap L_{jk}$ . As  $k + j \neq 0$ , we have  $|L_{jk} : (K \cap L_{jk})| < \infty$ , and Lemma 2.5 yields  $|G : K| < \infty$ .

(2) Suppose that  $i + j + k \neq 0$ , and let us first assume that  $j + k - 2i \neq 0$ . Observe that  $\theta^2(g^i)\theta(g^i)g^i \cdot \theta(g^{j-i})g^{k-i} = \theta^2(g^i)\theta(g^j)g^k \in X$ , for all  $g \in G$ . In

particular, if we set  $L = \{\theta^2(g^i)\theta(g^i)g^i\}_{g \in G}$ , then  $\theta(g^{j-i})g^{k-i} \in LX$ . Thus, in the notation of part (1), we have  $K_{j-i, k-i} \subseteq LX$ , and hence  $G/LX$  is finite since  $j+k-2i \neq 0$ . Next, if  $l \in L$ , then  $\theta(l) = l$ , so  $\theta^2(l^i)\theta(l^j)l^k = l^{i+j+k} \in X \cap L$ . Thus  $i+j+k \neq 0$  implies that the group  $L/(X \cap L)$  is finite and, by Lemma 2.5,  $|G : X| < \infty$  in this case. Finally, since  $\theta^2(\theta(g)^i)\theta(\theta(g)^j)\theta(g)^k = \theta^2(g^j)\theta(g^k)g^i$ , we see that  $X = \theta(X) = \{\theta^2(g^j)\theta(g^k)g^i\}_{g \in G}$ . Thus, by the above,  $|G : \theta(X)| < \infty$  if  $i+k-2j \neq 0$ . Hence  $|G : X| < \infty$  unless we have both  $i+k-2j = 0$  and  $j+k-2i = 0$ . Since these two equations are equivalent to  $i = j = k$ , the result follows.

**Lemma 2.8.** *Let  $i, j, l, n$  be some integers, and let  $\gamma$  be an automorphism of  $P$  such that  $\gamma^n(x) = x^{p^l}$ . If  $H = \{\gamma(x^i)x^j\}_{x \in P^*}$ , then  $x^{p^l i^n - (-j)^n} \in H$ .*

Proof. Note that  $H$  is  $\gamma$ -stable since  $\gamma(\gamma(x^i)x^j) = \gamma(\gamma(x)^i)\gamma(x)^j \in H$ . We use induction on  $k \leq n$  to show that

$$(*) \quad \gamma^{n-k}(x^{i^{n-k}(-j)^k}) \equiv x^{p^l i^n} \pmod{H},$$

where  $a \equiv b \pmod{H}$  indicates that  $ab^{-1} \in H$ , for  $a, b \in P^*$ . The lemma will follow by setting  $k = n$ .

If  $k = 0$ , then  $(*)$  has the shape  $\gamma^n(x^{i^n}) \equiv x^{p^l i^n}$  which is obtained from  $\gamma^n(x) = x^{p^l}$  by substituting  $x^{i^n}$  for  $x$ . Now suppose that  $(*)$  is true for some  $k < n$ . Since  $\gamma^{n-k}(a) = \gamma^{n-k-1}(\gamma(a))$  for  $a \in P^*$ , we have

$$\begin{aligned} x^{p^l i^n} &\equiv \gamma^{n-k}(x^{i^{n-k}(-j)^k}) = \gamma^{n-k-1}(\gamma(x^{i^{n-k}(-j)^k})) \\ &= \gamma^{n-k-1}(\gamma(x^i)^{i^{n-k-1}(-j)^k}) \equiv \gamma^{n-k-1}((x^{-j})^{i^{n-k-1}(-j)^k}) \\ &= \gamma^{n-k-1}((x^{i^{n-k-1}(-j)^{k+1}})) \pmod{H}, \end{aligned}$$

as desired.

**Proposition 2.9.** *Let  $G$  be a quasi-simple infinite periodic Chevalley group defined over a field of characteristic  $p > 0$ , and  $V$  be a non-trivial rational irreducible  $\bar{\mathbf{F}}_p G$ -module. Let  $F$  denote the character field of  $V$ , and let  $0 \neq v \in V$  be a highest weight vector. If  $H = \{0 \neq f \in \bar{\mathbf{F}}_p \mid gv = fv \text{ for some } g \in G\}$ , then  $H$  is a subgroup of finite index in  $F^*$ .*

Proof. If  $\phi$  is the representation of  $G$  afforded by  $V$ , then by Lemma 2.3,  $G$  can be realized over  $F$ . Let  $P_\theta$  denote the field of  $\theta$ -fixed elements in  $P$ . By the definition of  $\eta$ , we have  $H = \eta(G_v) = \eta(T_G)$ , and therefore,  $H \subseteq F^*$ . We show in fact that  $H$  has finite index in the multiplicative group of a larger field, namely either  $P^*$  or  $P_\theta^*$ . To this end, let  $\lambda = \sum_i a_i \omega_i$  be the highest weight of  $V$ , and recall that all  $a_i$  are non-negative integers.

(1) Suppose first that  $\theta = \text{Id}$ , so that  $P_\theta = P$ . Then  $H = \eta(T_G) = \{\prod_{i=1}^r t_i^{a_i}\}_{t_i \in P^*}$ , and  $|P^* : H| < \infty$ , by Lemma 2.4, since  $P^*$  is locally cyclic.

(2) Now let  $\theta^k = 1$  (where  $k = 2$  or  $3$ ), and let  $N(x) = \prod_i \theta^i(x)$  be the norm of  $x \in P$ . As is well-known,  $N(x) \in P_\theta$  if  $P$  is finite, and clearly this is also true for  $P$  locally finite. Therefore, if  $j \neq 0$  is any integer, then  $\{N(x^j)\}_{x \in P^*} = \{y^j\}_{y \in P_\theta^*}$  is a subgroup of finite index in  $P_\theta^*$ , by Lemma 2.4.

(3) Assume that  $\theta^2 = 1$ . By Lemma 2.2, it suffices to show that  $\{\theta(x^i)x^j\}_{x \in P}$  is a subgroup of finite index in either  $P^*$  or in  $P_\theta^*$  provided that  $i, j$  are non-negative

integers not both equal to 0. If  $i = j$ , this follows from (2). On the other hand, if  $i \neq j$ , then this claim follows from Lemma 2.6.

(4) Let  $\theta^3 = 1$ . By Lemma 2.2, it suffices to show that  $\{\theta^2(x^i)\theta(x^j)x^k\}_{x \in P}$  is a subgroup of finite index in either  $P^*$  or in  $P_\theta^*$  provided  $i, j, k$  are non-negative integers not all equal to 0. If  $i = j = k$ , then this follows from (2). On the other hand, if  $i, j$  and  $k$  are not all equal, then by Lemma 2.7, the desired fact holds unless  $i + j + k = 0$ . However, the latter cannot occur since  $i, j, k \geq 0$ .

(5) Finally, if  $G$  is of type  ${}^2B_2$ ,  ${}^2G_2$ , or  ${}^2F_4$ , then the result follows from Lemmas 2.2 and 2.8.

**Lemma 2.10.** *Let  $G$  be an infinite quasi-simple, locally finite group of Lie type, and let  $\phi : G \rightarrow GL(n, F)$  be a rational representation. If  $S_p$  is a Sylow  $p$ -subgroup of  $G$ , then  $\phi(S_p)$  is connected in the Zariski topology of  $M(n, F)$ .*

*Proof.* Let  $S^0$  denote the connected component of the identity in  $S_p$ . Then the index  $k = |S_p : S^0|$  is finite, and clearly  $S^0$  is a normal subgroup of  $N_G(S_p)$ . Thus  $S_p$  contains a  $N_G(S_p)$ -stable subgroup of finite index  $k \geq 1$ , and this is known to imply that  $k = 1$  since the action of  $N_G(S_p)$  on  $S_p$  is well-controlled.

The lemma can also be deduced from [9, Theorem 7].

### 3. REPRESENTATIONS OF QUASI-SIMPLE LOCALLY FINITE GROUPS

It is well-known that each quasi-simple infinite periodic linear group  $G$  is isomorphic to a Chevalley group over a locally finite field, and we use  $P$  to denote the definition field of  $G$ .

If  $\phi : G \rightarrow GL(n, \bar{\mathbf{F}}_p)$  is an irreducible representation, then we cannot immediately use the results above since  $\phi$  is not necessarily rational. As an example, take  $G = SL(2, P)$  and suppose that  $P$  contains a subfield  $P_0$  with  $|P : P_0| = 2$ . Let  $\sigma$  be a non-trivial Galois automorphism of  $P/P_0$  and let  $\nu$  be the natural representation of  $G$  (i.e. the one given by matrices of size 2). Then the representation  $\sigma\nu$ , the twist of  $\nu$  by  $\sigma$ , is not rational and has no highest weight. Here  $\sigma\nu$  is quasi-equivalent to  $\nu$ , but  $\nu \otimes \sigma\nu$  is not quasi-equivalent to any rational representation of  $G$ .

The following theorem was proved by Borel and Tits [2] for groups different from  ${}^2B_2, {}^2G_2, {}^2F_4$ . The remaining three types were handled in [6, Theorem 3.5]. This was later generalized by Seitz [9]; in addition, the latter paper mentions an unpublished proof by J. Tits.

**Theorem 3.1.** *Let  $\phi$  be an irreducible representation of  $G$ . Then  $\phi$  can be expressed as a finite tensor product  $\otimes_i \sigma_i \phi_i$ , where the  $\phi_i$  are infinitesimally irreducible representations of  $G$  and the  $\sigma_i$  are field automorphisms of  $P$ .*

One can express this by saying that  $\phi$  is a tensor product of representations that are quasi-equivalent to infinitesimally irreducible ones. This result is a natural generalization of the famous description of the irreducible representations of finite Chevalley groups. If  $G$  is finite, then  $\sigma_i$  coincides with  $f^{n_i}$  for some  $n_i \in \mathbf{N}$ , where  $f : x \mapsto x^p$  for  $x \in P$  extends to  $G$ , as described in Section 2.

In spite of the absence of a highest weight for non-rational  $\phi$ , one can still define the highest vector to be a non-zero vector fixed by a Sylow  $p$ -subgroup  $S$  of  $G$ . To justify this, observe that  $G$  contains an irreducible finite Chevalley subgroup  $G_0$  of the same type as  $G$  and with  $S_p(G_0) \subseteq S$ . Thus there is a unique line

$\langle v \rangle \subset V$  fixed elementwise by  $S_p(G_0)$ , and then  $v$  is fixed by  $S$ , since  $\phi(S)$  can be upper-triangularized.

Let  $v$  be a highest vector for  $\phi$  and let  $G_v$  denote the stabilizer of  $v$  in  $G$ . As in the finite group case, the action of  $G_v$  on  $\langle v \rangle$  defines a linear character  $\eta : G_v \rightarrow \bar{\mathbf{F}}_p$  which we call the highest vector character. Observe that  $v = \otimes_i v_i$  where, for each  $i$ ,  $v_i$  is a highest vector of  $\phi_i$ . It follows from Theorem 3.1 that  $\eta = \chi_1^{\sigma_1} \cdots \chi_k^{\sigma_k}$  where  $\chi_1, \dots, \chi_k$  are the highest vector characters of infinitesimally irreducible representations of  $G$  and  $\chi_i^{\sigma_i}(g) = \sigma_i(\chi_i(g))$  for all  $g \in G_v$ .

**Lemma 3.2.** *Let  $\phi$  and  $\phi'$  be non-equivalent irreducible representations of  $G$  over  $\bar{\mathbf{F}}_p$ . Then there exists a finite subgroup  $H$  of  $G$ , of the same Lie type as  $G$ , such that  $\phi|_H$  and  $\phi'|_H$  are both irreducible,  $\phi|_H$  is not equivalent to  $\phi'|_H$ , and  $\eta|_H \neq \eta'|_H$ .*

*Proof.* To start with,  $G$  is a union of a tower of finite groups  $G_1 \subset G_2 \subset \dots$  of the same type as  $G$ . By Burnside's theorem, the irreducibility of  $\phi$  is equivalent to the existence in  $\phi(G)$  of  $n^2$  linear independent matrices. Hence, there exists  $m$ , with  $\phi|_{G_i}$  and  $\phi'|_{G_i}$  both irreducible for all  $i \geq m$ . Furthermore, since  $\phi$  and  $\phi'$  are non-equivalent irreducible representations of  $G$ , it follows easily that  $\phi|_{G_i}$  and  $\phi'|_{G_i}$  are not equivalent for some  $i \geq m$ , and hence their Brauer characters are distinct. By Lemma 2.1,  $\eta|_H \neq \eta'|_H$  for  $H = G_i$ .

**Lemma 3.3.** *Let  $\phi : G \rightarrow GL(n, \bar{\mathbf{F}}_p)$  be an irreducible representation with character  $\chi$ . If  $F = \mathbf{F}_p(\chi)$  and let  $K = \mathbf{F}_p(\eta)$  denote the fields generated by the values  $\chi(g)_{g \in G}$  and  $\eta(g)_{g \in G_v}$ , respectively, then  $F = K$ .*

*Proof.* Observe that  $K \subseteq F$ , since  $\phi(G)$  can be realized over the character field (Lemma 2.3). Thus, we can assume that  $G \subseteq GL(n, F)$  and that  $V = F^{(n)}$  is the natural module for  $GL(n, F)$ . Suppose that  $K \neq F$ . Then there exists a non-trivial automorphism  $\sigma$  of  $\mathbf{F}_p(\chi)$  which acts trivially on  $\mathbf{F}_p(\eta)$ . As absolutely irreducible representations of  $G$  are determined by their characters and  $\chi^\sigma \neq \chi$ , we see that the representations associated with  $\chi$  and  $\chi^\sigma$  are not equivalent.

Let  $v \in V$  be the highest weight vector of  $\phi$ . By replacing  $G$  by a conjugate, if necessary, we can assume that  $v = (1, \dots, 0)^t \in F^{(n)}$ , where  $t$  stands for the transpose. Now  $\sigma$  acts on  $V$ , and note that  $\sigma(v) = v$ . Therefore,  $G_v$  and  $G_v^\sigma$  both stabilize  $v$ . Let  $\eta_1$  denote the character of the action of  $G_v^\sigma$  on  $\langle v \rangle$ . Then, with respect to the natural basis  $\{v, v_2, \dots, v_n\}$ , the matrices of  $g$  and  $g^\sigma$ , for  $g \in G_v$ , have the shape  $\begin{pmatrix} \eta(g) & * \\ 0 & * \end{pmatrix}$  and  $\begin{pmatrix} \eta_1(g^\sigma) & * \\ 0 & * \end{pmatrix}$ , so it follows that  $\eta_1(g^\sigma) = (\eta(g))^\sigma$ .

This means that the correspondence  $\phi \mapsto \eta$  is  $\sigma$ -invariant. By Lemma 3.2, there exists a finite subgroup  $H$  of  $G$ , of the same Lie type as  $G$ , such that  $\phi|_H$  is not equivalent to  $\phi^\sigma|_H$ . Let  $\bar{G}$  be the algebraic group defining  $G$  as a Chevalley group. Set  $\tau = \phi|_H$ ,  $\tau_1 = \phi^\sigma|_H$ , and let  $\rho$  and  $\rho_1$  be rational representations of  $\bar{G}$  such that  $\rho|_H = \tau$  and  $\rho_1|_H = \tau_1$ . (In general, it is not true that  $\rho|_G = \phi$  and  $\rho_1|_G = \phi^\sigma$ .)

Of course,  $\tau_1 = \tau^\sigma$ , and we let  $F_H$  be the character field for  $\tau$ . As  $\sigma(F_H) = F_H$ , it follows that  $F_H$  is also a character field for  $\tau^\sigma$ . Moreover, since  $F_H$  is finite,  $\sigma$  can be expressed as a power of the Frobenius automorphism  $f$  of  $F_H$ , and thus  $\tau_1 = \tau^{f^k}$  for some  $k$ . Replacing  $\rho$  by an equivalent representation, if necessary, we can assume that  $f(\rho(\bar{G})) = \rho(\bar{G})$ , and that, for each algebraic subgroup  $X$  of  $\bar{G}$  defined over  $\mathbf{F}_p$ , we have  $f(\rho(X)) = \rho(X)$ . In particular, since  $\bar{G}_v = \{g \in \bar{G} \mid gv \in \langle v \rangle\}$  is algebraic and defined over  $\mathbf{F}_p$ ,  $f$  preserves  $\bar{G}_v$ . Also, as we saw in Section 2,  $f$  preserves  $H$ . Hence  $f$  preserves  $H_v = H \cap \bar{G}_v$ , and therefore  $H_v^\sigma = f^k(H_v) = H_v$ . It now follows



from Lemma 2.1 that  $\tau$  is equivalent to  $\tau_1$ , which is a contradiction. Thus,  $K = F$ , as required.

Let  $F$  be a field of characteristic  $p$  and let  $f$  denote the Frobenius automorphism of  $F$  (i.e.,  $f(x) = x^p$  for  $x \in F$ ). If  $\alpha, \beta$  are automorphisms of  $F$ , we say that  $\alpha, \beta$  are equivalent if  $\alpha = f^l \circ \beta$  for some integer  $l$ , and let us write  $\tilde{\alpha}$  for the equivalence class of  $\alpha$ . Note that each such class contains countably many members. The following is [9, Lemma 2.1].

**Lemma 3.4.** *Let  $\alpha_1, \dots, \alpha_m$  be non-equivalent automorphisms of an infinite field  $F$ . If  $q \in F[x_1, \dots, x_m]$  is a polynomial such that  $q(\alpha_1(x), \dots, \alpha_m(x)) = 0$  for all  $x \in F$ , then  $q = 0$ .*

As a consequence, we have

**Lemma 3.5.** *Let  $F$  be a locally finite field of characteristic  $p$  and suppose that  $\alpha_1, \alpha_2, \dots, \alpha_n$  are field automorphisms. For any positive integers  $a_1, a_2, \dots, a_n$ , let  $H$  be the subgroup of  $F^\bullet$  given by*

$$H = \{ \alpha_1(x)^{a_1} \alpha_2(x)^{a_2} \cdots \alpha_n(x)^{a_n} \mid x \in F^\bullet \}.$$

*If  $L$  is the subfield of  $F$  generated by  $H$ , then  $|F : L| < \infty$ , and  $\{ \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n \}$  is a union of orbits under  $\Sigma = \text{Gal}(F/L)$  (where  $\sigma \in \Sigma$  maps  $\alpha_i$  to  $\sigma \circ \alpha_i$ ).*

*Proof.* It suffices to assume that  $\alpha_1, \alpha_2, \dots, \alpha_n$  are non-equivalent. Indeed, if  $\alpha_i = f^k \circ \alpha_j$  with  $k \geq 0$ , then  $\alpha_i(x) = f^k \circ \alpha_j(x) = \alpha_j(x)^{p^k}$ , so we can replace  $\alpha_i(x)$  by  $\alpha_j(x)^{p^k}$  in the definition of  $H$ .

Let  $\sigma \in \text{Gal}(F/L)$ . Then for all  $x \in F$  (including  $x = 0$ ), we have

$$\begin{aligned} \alpha_1(x)^{a_1} \alpha_2(x)^{a_2} \cdots \alpha_n(x)^{a_n} &= \sigma(\alpha_1(x)^{a_1} \alpha_2(x)^{a_2} \cdots \alpha_n(x)^{a_n}) \\ &= \beta_1(x)^{a_1} \beta_2(x)^{a_2} \cdots \beta_n(x)^{a_n}, \end{aligned}$$

where  $\beta_i = \sigma \circ \alpha_i$ . Let  $A = \{ \alpha_1, \alpha_2, \dots, \alpha_n \}$ ,  $B = \{ \beta_1, \beta_2, \dots, \beta_n \}$ , and set  $S = A \cup B$ . Note that the elements of  $A$  are non-equivalent, so the same is true of the elements of  $B$ . Thus,  $|\tilde{A}| = |\tilde{B}| = n$  and, if  $\tilde{A} \neq \tilde{B}$ , then by introducing a variable  $x_r$  for each distinct element of  $\tilde{S}$ , the above displayed equation yields a nontrivial polynomial in the automorphisms which annihilates  $F$ , contradicting Lemma 3.4. Thus, we must have  $\tilde{A} = \tilde{B}$ , so that  $\Sigma = \text{Gal}(F/L)$  permutes  $\tilde{A}$ . In particular, since each equivalence class is countable, it follows that  $\Sigma$  is countable. But, as is well-known,  $|F : L| = \infty$  implies that  $\text{Gal}(F/L)$  is uncountable, so we conclude that  $|F : L| < \infty$ .

**Lemma 3.6.** *If  $P$  is the defining field of  $G$ , then the character field  $F$  of an irreducible representation  $\phi$  of  $G$  has finite index in  $P$ .*

*Proof.* By Lemma 2.3 (or 3.3),  $\eta(G_v) \subseteq F$ , and therefore it suffices to prove that the field  $\langle \eta(G_v) \rangle$  has finite index in  $P$ . Recall that  $\eta(G_v) = \eta(T_G)$  and that  $\eta|_{T_G} = \chi_1^{\sigma_1} \cdots \chi_k^{\sigma_k}$ , where  $\sigma_1, \dots, \sigma_k$  are automorphisms of  $P$ . By Lemma 2.2, each  $\chi_i$  has shape  $\chi_i(x) = \prod_j \gamma_j(x)^{a_j}$  for some nonnegative integers  $a_j$ . Therefore,  $\eta|_{T_G}$  has a similar shape, and it suffices to prove that the field  $P_0 = \langle \{ \alpha_1(x) \dots \alpha_n(x) \}_{x \in P} \rangle$  has finite index in  $P$ , where  $\alpha_1, \dots, \alpha_n$  are any field automorphisms of  $P$ . This, of course, follows from Lemma 3.5.

## 4. GROUP ALGEBRAS

If a group  $H$  acts on a group  $G$  via automorphisms, then we denote by  $D_G(H)$  the subgroup of all elements of  $G$  whose  $H$ -orbit is finite. If  $H = G$  we assume that  $G$  acts on itself via inner automorphisms, so  $D_G(G)$  is the union of the finite conjugacy classes of  $G$ . If  $H$  is a subgroup of  $G$ , then  $H$  acts on  $G$  by conjugation and we set  $D_G(H) = \{g \in G \mid |\{hgh^{-1}\}_{h \in H}| < \infty\}$ . Equivalently,  $D_G(H) = \{g \in G \mid |H : C_H(g)| < \infty\}$ . It is clear that  $D_G(H)$  is a subgroup of  $G$ . As usual, we let  $KG$  denote the group algebra of  $G$  over the field  $K$ . For convenience, we quote the following corollary of [8, Theorem 4.2.9].

**Lemma 4.1.** *Let  $G$  be a group acting by automorphisms on a group  $A$ , and let  $J_1, \dots, J_n$  be  $G$ -stable nonzero ideals of  $KA$  with  $J_1 \cdots J_n = \{0\}$ . Then  $A$  contains a non-identity element with a finite  $G$ -orbit.*

As a consequence, we have

**Lemma 4.2.** *Let  $G$  act as automorphisms on the infinite abelian periodic group  $H$  and let  $G_1$  be a normal subgroup of  $G$  of finite index. If  $KH$  has no proper non-zero  $G$ -stable ideal other than the augmentation ideal, then  $KH$  has no proper non-zero  $G_1$ -stable ideal other than the augmentation ideal.*

Proof. Let  $\{g_1, \dots, g_n\}$  be a transversal for  $G_1$  in  $G$ , and let  $I \neq \{0\}$  be a proper  $G_1$ -stable ideal of  $KH$  different from the augmentation ideal. Put  $I_j = g_j(I)$  and let  $J = \bigcap_j I_j$ . Then  $GJ = J$ , and since  $J$  is clearly not the augmentation ideal, the hypothesis implies that  $J = \{0\}$ . Thus  $I_1 \cdots I_n = \{0\}$ , and note that each  $I_j$  is  $G_1$ -stable since  $G_1$  is normal in  $G$ . Thus, by Lemma 4.1,  $H$  has a non-trivial finite  $G_1$ -orbit and hence a nontrivial finite  $G$ -orbit. Since  $H$  is locally finite, this implies that  $H$  has a non-identity finite  $G$ -stable subgroup  $N$ . Also  $N$  is normal in  $H$ , since  $H$  is abelian, and  $N$  is different from  $H$ , since  $H$  is infinite. The augmentation ideal of  $KN$  extended to  $KH$  is therefore a non-zero  $G$ -stable ideal of  $KH$  different from the augmentation ideal, and this is a contradiction.

In addition, we quote a corollary to [11, Theorem 1], see also [8, Section 8.4].

**Lemma 4.3.** *Let  $G$  act on an abelian group  $A$  in such a way that the  $n$ -fold commutator satisfies  $[\cdots[[A, G], G] \cdots] = \{1\}$ . Assume further that  $D_A(G) = C_A(G)$ . If  $I \neq \{0\}$  is a  $G$ -stable ideal of  $KA$ , then  $I \cap KC_A(G) \neq \{0\}$ .*

Next, we need

**Lemma 4.4.** *Let  $F$  be an infinite locally finite field of characteristic  $p > 0$  and let  $A = F^+$  be the additive group of  $F$ . Suppose  $B$  is a finite subgroup of  $A$ , and let  $K$  be an algebraically closed field of characteristic  $\neq p$ . If  $\chi : A \rightarrow K^*$  and  $\kappa : B \rightarrow K^*$  are irreducible characters of  $A$  and  $B$ , respectively, with  $\chi \neq 1_A$ , then there exists  $f \in F^*$  such that  $\chi^f|_B = \kappa$ . Here,  $\chi^f$  denotes the character defined for  $a \in A$  by  $\chi^f(a) = \chi(fa)$ .*

Proof. Since  $F$  is an infinite locally finite field,  $B$  is finite and  $\chi \neq 1_A$ , there exists a finite subfield  $L$  of  $F$  with  $B \subseteq L$ ,  $\chi|_{L^+} \neq 1_{L^+}$  and such that  $\kappa$  extends to a non-principal character  $\mu : L^+ \rightarrow K^*$ . Furthermore, since  $L^*$  is clearly transitive on the non-principal characters of  $L^+$ , there exists  $f \in L^*$  with  $(\chi|_{L^+})^f = \mu$ , and hence  $\chi^f|_B = \kappa$ .

The following is an easily handled particular case of [3, Theorem 1].

**Lemma 4.5.** *Let  $F$  be an infinite locally finite field of characteristic  $p > 0$  and let  $A = F^+$  be the additive group of  $F$ . Suppose  $K$  is a field of characteristic  $\neq p$ , and extend the action of the multiplicative group  $M = F^*$  on  $F^+ = A$  to  $KA$ . If  $0 \neq I \subset KA$  is a proper  $M$ -stable ideal, then  $I$  coincides with the augmentation ideal of  $KA$ .*

*Proof.* It suffices to assume that  $K$  is algebraically closed. Suppose that  $I$  is not the augmentation ideal. Then there exists a finite subgroup  $B$  of  $A$  such that  $I_B = I \cap KB \neq \{0\}$  and  $I_B$  is not the augmentation ideal of  $KB$ . Replacing  $B$  by a larger group if necessary, we can assume that  $B$  is just  $\mathbf{F}_q^+$  for some finite subfield  $\mathbf{F}_q$  of  $F$ . Then  $I_B$  is stable under  $\mathbf{F}_q^*$ . As  $I_B$  is not the augmentation ideal of  $KB$ , there exists an irreducible representation  $\chi \neq 1_B$  of  $KB$  such that  $\chi(I_B) = \{0\}$ . If we define  $\chi^k$  with  $k \in \mathbf{F}_q^*$  to be the character  $\chi^k(b) = \chi(kb)$  for all  $b \in B = \mathbf{F}_q^+$ , then  $\chi^k(I_B) = \{0\}$  for all  $\chi^k$ . Observe that  $\chi^k$  runs over all the irreducible representations of  $B$  other than  $1_B$  as  $k$  runs over  $\mathbf{F}_q^*$  because the group  $\mathbf{F}_q^*$  acts transitively on the set  $\mathbf{F}_q^+ \setminus \{0\}$ . Thus  $\chi(I_B) = \{0\}$  implies that  $\dim I_B = 1$ . Finally, let  $L$  be a finite subfield of  $F$  properly containing  $\mathbf{F}_q$ . Then  $\dim I_L = 1$ . But  $I_L \supseteq I_B \cdot KL^+$ , so  $\dim I_L \geq \dim(I_B \cdot KL^+) = (\dim I_B) \cdot |L^+ : B| > 1$ , and this is the required contradiction.

**Theorem 4.6.** *Let  $G$  be an infinite quasi-simple locally finite group of Lie type with defining characteristic  $p > 0$ . Suppose  $\phi : G \rightarrow GL(V)$  is an absolutely irreducible rational representation of  $G$ , where  $V$  is a vector space finite dimensional over a field  $F$ . Assume that  $\phi$  cannot be realized over any subfield of  $F$  (equivalently,  $F$  is the field of the character of  $\phi$ ). If  $A = V^+$  is the additive group of  $V$  and if  $K$  a field of characteristic different from  $p$ , then the group algebra  $KA$  of  $A$  over  $K$  has no proper non-zero  $G$ -stable ideal other than the augmentation ideal.*

*Proof.* Let  $I$  be a proper non-zero  $G$ -stable ideal of  $KA$ , let  $S$  denote a Sylow  $p$ -subgroup of  $G$ , and let  $B = C_A(S)$ . It follows from Lemma 2.10 that  $B = D_A(S)$ . Thus, since  $S$  can be upper triangularized, we conclude from Lemma 4.3 that  $J = I \cap KB \neq 0$ . According to Section 3,  $B$  is isomorphic to  $F^+$ . If  $N = N_G(S)$ , then  $B$  is  $N$ -stable, so  $J$  is  $N$ -stable as well. As we know, the action of  $N$  on  $B$  is described by the linear character  $\eta$  and, according to Proposition 2.9, the group  $\eta(N)$  is of finite index in  $F^*$ . By Lemma 4.5,  $KB$  has no proper non-zero  $F^*$ -stable ideal other than the augmentation ideal and consequently, by Lemma 4.2, the augmentation ideal is the only  $N$ -stable proper non-zero ideal of  $KB$ . Therefore,  $J$  is the augmentation ideal of  $KB$ , so  $b - 1 \in J \subset I$  for each  $b \in B$ , and thus  $gbg^{-1} - 1 \in I$  for each  $g \in G$ . In particular, if  $H = \{a \in A \mid a - 1 \in I\}$ , then  $H$  is a non-identity  $G$ -stable subgroup of  $A$  containing all  $gBg^{-1} \cong F^+$ . As  $G$  is  $F$ -irreducible, we conclude that  $H = A$  and hence that  $I$  is the augmentation ideal of  $KA$ .

**Proof of Theorem 1.1** Let  $E$  denote the commuting ring of the irreducible  $FG$ -module  $V$ . Then  $E$  is a field and  $|E : F| < \infty$ . Furthermore,  $V$  is absolutely irreducible as an  $EG$ -module with the same  $G$ -action. Therefore, the result follows from Theorem 4.6.

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