CONSTRUCTION OF FREE SUBGROUPS IN THE GROUP OF UNITS OF MODULAR GROUP ALGEBRAS

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Abstract. Let KG be the group algebra of a p'-group G over a field K of characteristic $p > 0$, and let $U(KG)$ be its group of units. If KG contains a nontrivial bicyclic unit and if K is not algebraic over its prime field, then we prove that the free product $\mathbb{Z}_p * \mathbb{Z}_p * \mathbb{Z}_p$ can be embedded in $U(KG)$.

1. Introduction

Let KG be the group algebra of the group G over the field K, and let $U(KG)$ be its group of units. Motivated by the work of Pickel and Hartley [4], and Sehgal ([7, pg. 200]) on the existence of free subgroups in the integral group ring $\mathbb{Z}G$, analogous conditions for $U(KG)$ have been intensively investigated in [1], [2] and [3].

Recently Marciniak and Sehgal [5] gave a constructive method for producing free subgroups in $U(\mathbb{Z}G)$, provided $\mathbb{Z}G$ contains a nontrivial bicyclic unit. In this paper we prove an analogous result for the modular group algebra KG, whenever K is not algebraic over its prime field $GF(p)$. Specifically, if $\mathbb{Z}_p$ denotes the cyclic group of order p, then we prove:

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Theorem. *Let K be a field of characteristic $p > 0$ containing an element t transcendental over its prime subfield. Let G be a group which has two elements x, y such that x has finite order n, y does not normalize $\langle x\rangle$, and the subgroup $\langle x, y^{-1}xy\rangle$ has no p-torsion. If we let*

$$a = (1-x)y\widehat{x}, \quad b = \widehat{x}y^{-1}(1-x^{\delta}), \quad \widehat{x} = \sum_{i=0}^{n-1} x^i,$$

where $\delta = (-1)^p$, then $U = U(KG)$ contains

$$\langle 1+ta,\ 1+tbab,\ 1+t(1-b)aba(1+b)\rangle \cong \mathbb{Z}_p * \mathbb{Z}_p * \mathbb{Z}_p.$$

The assumption that $\langle x, y^{-1}xy\rangle$ has no p-torsion cannot be replaced by the weaker statement $\gcd(n,p) = 1$. Indeed, let $p > 2$ and let $G = P \rtimes X$ be a finite dihedral group of order $2p$, where P is cyclic of order p and X has order 2. Certainly X is not normal in G and $|X| = 2$ is relatively prime to p. Now let $\operatorname{char} K = p$ and let I be the kernel of the natural epimorphism $KG \to K(G/P) \cong KX$. Then it is easy to see that $I^p = 0$. Furthermore, since KX is commutative, it follows that $U' \subseteq 1 + I$. But then U' is a nilpotent group of period p, and hence U cannot contain an isomorphic copy of $\mathbb{Z}_p * \mathbb{Z}_p$. Similarly, when $p = 2$, take $G = P \rtimes X \cong \mathrm{Alt}_4$, where $|P| = 4$ and $|X| = 3$.

Corollary. *If G is a nonabelian torsion p'-group and K is not algebraic over its prime subfield $GF(p)$, then $U(KG)$ contains a free noncyclic group.*

2. The Proofs

Let A be an F-algebra, and let $A[t]$ denote the polynomial ring over A in the variable t. We start with:

Lemma. *Assume that* $\operatorname{char} F = p > 0$, *and let a and b nonzero elements of A such that $a^2 = b^2 = 0$ and ba is not nilpotent. Let us define*

$$\begin{aligned} x_1 &= 1 + ta \\ x_2 &= 1 + tbab \\ x_3 &= 1 + t(1-b)aba(1+b). \end{aligned}$$

Then, x_1, x_2 and x_3 are units of order p in $A[t]$, and

$$\langle x_1, x_2, x_3\rangle \cong \mathbb{Z}_p * \mathbb{Z}_p * \mathbb{Z}_p.$$

Proof. Note that $x_1^p = x_2^p = x_3^p = 1$, since a, bab and $(1-b)aba(1+b)$ all have square zero. We now show that x_1, x_2 and x_3 generate $\mathbb{Z}_p * \mathbb{Z}_p * \mathbb{Z}_p$.

Suppose not. Then there exists an identity of the form

$$(*) \qquad x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_n}^{j_n} = 1,$$

with $i_s \in \{1,2,3\}$, with no two neighboring indices being equal, and with exponents $1 \le j_s \le p-1$. If some $i_s = 2$, then after cyclic rotation, we may suppose that $i_1 = 2$. On the other hand, if no $i_s = 2$, then we can conjugate equation $(*)$ by x_2 and again suppose that $i_1 = 2$. In other words, without loss of generality, we may assume that $i_1 = 2$.

An easy induction now shows that any partial product $x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_k}^{j_k}$ of the left-hand side of $(*)$ with $1 \le k \le n$ is a polynomial in $A[t]$ of degree ≥ 1 with leading coefficient equal to $\pm j_1 j_2 \cdots j_k \, c_k$, where

$$c_k = \begin{cases} (ba)^r & \text{if } i_k = 1, \\ (ba)^r b & \text{if } i_k = 2, \\ (ba)^r(1+b) & \text{if } i_k = 3, \end{cases}$$

for some $r \ge 1$. Note that all such leading coefficients are not zero since $1 \le j_s \le p-1$ and since either c_k or $c_k a$ is a power of the nonnilpotent element ba. In particular, when $k = n$, equation $(*)$ yields a contradiction. □

Proof of the Theorem. We apply the Lemma with $F = GF(p)$ and with $A = GF(p)G$. Since $t \in K$ is transcendental over F, it follows that $KG \supseteq A[t]$. Now $a^2 = b^2 = 0$, so it is enough to show that ba is not nilpotent. We consider two cases:

(i) $p = 2$. Here we have

$$ba = \widehat{x} y^{-1} (1 + x^2) y \widehat{x} = \widehat{x}(1 + z^2)\widehat{x},$$

where we set $z = y^{-1}xy$. If ba is nilpotent, then by [6, Theorem 2.3.4], its trace must be zero since $\langle x, z \rangle$ is a p'-group by assumption. Now the above product is the sum of the two terms $\widehat{x}\,\widehat{x}$, $\widehat{x} z^2 \widehat{x}$, and we claim that the only contribution to the trace comes from $\widehat{x}\,\widehat{x}$ and is equal to n.

Indeed, if $x^i z^2 x^j = 1$, then we have that $z^2 = x^{-i-j} \in \langle x \rangle$. But z has odd order, so $\langle z \rangle = \langle z^2 \rangle$, and hence $z \in \langle x \rangle$, contrary to the hypothesis. Thus the trace of ba is equal to the trace of $\widehat{x}^2 = n\widehat{x}$ and this is equal to n, which is not zero in K.

(ii) $p \ne 2$. In this case we have

$$ba = \widehat{x} y^{-1} (2 - x - x^{-1}) y \widehat{x}.$$

As before, let us set $z = y^{-1}xy$. Then $ba = \widehat{x}(2 - z - z^{-1})\widehat{x}$, and if ba is nilpotent, then by [6, Theorem 2.3.4] again, its trace must be zero since

$\langle x, z\rangle$ is a p'-group. Here the above product is the sum of the three terms corresponding to $\widehat{x}2\widehat{x}$, $\widehat{x}z\widehat{x}$, $\widehat{x}z^{-1}\widehat{x}$, and we claim that the only contribution to the trace of ba comes from $\widehat{x}2\widehat{x}$, and is equal to $2n$.

Indeed, if $x^i z^\epsilon x^j = 1$ with $\epsilon = \pm 1$, then $z^\epsilon = x^{-i-j} \in \langle x\rangle$ and hence $z = y^{-1}xy \in \langle x\rangle$, contrary to the hypothesis. This proves the claim and, since $2n$ is not zero in K, the result follows. □

Proof of the Corollary. Suppose $U(KG)$ does not contain a noncyclic free subgroup. Then $U(KG)$ cannot contain $\mathbb{Z}_p * \mathbb{Z}_p * \mathbb{Z}_p$. In particular, since G is not abelian, the Theorem implies that all cyclic subgroups of G are normal and hence G is a Hamiltonian group. Thus $p \neq 2$ and $G = A \times E \times Q_8$, where A is an abelian group in which every element has odd order, E is an elementary abelian 2-group and Q_8 is the quaternion group of order 8. But then KQ_8 contains the direct summand $M_2(K)$, the full 2×2 matrix ring over K. Furthermore, $GL_2(K)$ contains nontrivial free subgroups (see for example [1]). Thus this situation cannot occur and the Corollary is proved. □

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