

FINITE PRESENTATION

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ABSTRACT

This paper surveys basic properties of finite presentation in groups, Lie algebras and rings. It includes some new results and also new, more elementary proofs, of some results that are already in the literature. In particular, we discuss examples of Stallings and of Roos on coherence and a recent theorem of Alahmadi and Alsulami on Morita invariance.

1. Introduction

In this paper we study finitely presented algebraic objects in four different contexts. Admittedly, we could work with more general universal algebras, but we prefer to keep things concrete. The objects of interest here are:

- i. groups,
- ii. Lie algebras over a field K ,
- iii. R -algebras where R is a ring, and
- iv. more general rings.

In case (iii), we mean rings S that contain R as a subring with the same 1 and are generated by R and its centralizer $\mathbb{C}_S(R)$ in S . Now with each of these

there are free objects, namely free groups, free Lie algebras, free R -algebras and free rings. Furthermore, there are homomorphisms and their kernels. Of course, kernels for groups are normal subgroups, while the others are Lie ideals and just plain ideals.

We say that one of these objects A is *finitely presented* if there exists a finitely generated free object F and an epimorphism $\pi: F \rightarrow A$ such that the kernel of π is finitely generated as a kernel, that is as either a normal subgroup or an ideal. Note that a finitely presented object is necessarily finitely generated.

The major topics considered here are: (1) Results of Baumslag [Bm] showing that a group G is finitely presented if and only if its group algebra $K[G]$ is and similarly that a K -Lie algebra L is finitely presented if and only if its universal enveloping algebra $U(L)$ is. (2) A number of variations of a construction of Abels [A] showing that certain finitely generated objects have infinitely generated centers. (3) An example of Stallings [Sta] of a finitely presented group that is not coherent and an example of Roos [Ro] of a finitely presented Lie algebra that is not coherent. Our arguments here are purely algebraic and hence markedly different from the original topological proofs. (4) Finite presentation is preserved by certain idealizers and a slightly simpler proof of a recent theorem of Alahmadi and Alsulami [AA2] showing that finite presentation is a Morita invariant.

We start with some basic results.

LEMMA 1.1: *Let A and B be finitely generated objects of the same type and let A be finitely presented. If $\vartheta: B \rightarrow A$ is an epimorphism, then the kernel of ϑ is finitely generated as a kernel.*

Proof. Since A is a finitely presented object, there exists a free object $F = \langle x_1, x_2, \dots, x_n \rangle$, with the x_i as free generators, and an epimorphism $\pi: F \rightarrow A$, such that $\ker \pi = J$ is finitely generated as a kernel. Clearly A is generated by the elements $a_i = \pi(x_i)$. Next, we are given the epimorphism $\vartheta: B \rightarrow A$, so we can choose $b_i \in B$ with $\vartheta(b_i) = a_i$. Since F is free on the x_i 's, we can define a homomorphism $\alpha: F \rightarrow B$ by $\alpha(x_i) = b_i$. Notice that the composition $\vartheta\alpha: F \rightarrow B \rightarrow A$ is equal to π . In particular, if $f \in F$, then $\alpha(f) \in \ker \vartheta$ if and only if $f \in \ker \pi$.

Now $B = \langle g_1, g_2, \dots, g_m \rangle$ is finitely generated, and since π is onto, we can choose $f_j \in F$ with $\vartheta(g_j) = \pi(f_j) = \vartheta\alpha(f_j)$. Thus $g_j - \alpha(f_j) \in \ker \vartheta$ for all j . Of course, in the context of groups, addition corresponds to multiplication and

subtraction to a version of division. Let I be the kernel in B generated by the finitely many elements $g_j - \alpha(f_j)$. Since each g_j is contained in $I + \text{Im}(\alpha)$, and since I is a kernel, we see that $B = I + \text{Im}(\alpha)$.

Finally, let $u \in \ker \vartheta$. Then $u \in I + \text{Im}(\alpha)$, so $u = v + \alpha(f)$ for some $v \in I$ and $f \in F$. Since $v \in I \subseteq \ker \vartheta$, we have $\alpha(f) \in \ker \vartheta$ and hence $f \in \ker \pi = J$. Thus since $\alpha(J) \subseteq \ker \vartheta$, we see that $\ker \vartheta = I + \alpha(J)$. Now I is finitely generated as a kernel and J is finitely generated as a kernel in F . Thus $\alpha(J)$ is finitely generated as a kernel in $\alpha(F) \subseteq B$. We conclude that $\ker \vartheta = I + \alpha(J)$ is finitely generated as a kernel in B , and the result follows. ■

The above efficient proof is based on the one in [RS]. Conversely we have

LEMMA 1.2: *Let A and B be finitely generated objects of the same type and let B be finitely presented. If $\vartheta: B \rightarrow A$ is an epimorphism and if the kernel of ϑ is finitely generated as a kernel, then A is finitely presented.*

Proof. Since B is finitely presented, there exists a finitely generated free object F and an epimorphism $\pi: F \rightarrow B$ so that the kernel of π is finitely generated as a kernel. Then the composite map $\vartheta\pi: F \rightarrow B \rightarrow A$ is an epimorphism and its kernel L in F satisfies $L \supseteq \ker \pi$ and $B = F/\ker \pi \supseteq L/\ker \pi = \ker \vartheta$. Since $\ker \vartheta$ is finitely generated as a kernel in B and $\ker \pi$ is finitely generated as a kernel in F , it is clear that L is finitely generated as a kernel in F . Hence A is finitely presented. ■

As a consequence, we obtain

LEMMA 1.3: *Let A be a finitely generated object and write $A = B \oplus C$, a direct sum of objects of the same type. Then B and C are finitely generated. Furthermore, if A is finitely presented, then so are B and C .*

Proof. We have an epimorphism $\pi: A \rightarrow B$ with kernel C and we have an epimorphism $\vartheta: A \rightarrow C$. Since A is finitely generated, the latter implies that C is finitely generated and hence finitely generated as a kernel. The previous lemma now yields the result.

Note that if A is a R -algebra or a ring, then C is an ideal of A and it is singly generated as an ideal by the central idempotent $e = 0 \oplus 1$, that is $C = Ae$. ■

Furthermore, we have

PROPOSITION 1.4: *Let A be a finitely presented object. Then all homomorphic images of A are finitely presented if and only if A satisfies the ascending chain condition on kernels.*

Proof. Of course A satisfies the ascending chain condition on kernels if and only if all kernels are finitely generated as kernels. Thus Lemmas 1.1 and 1.2 yield the result. ■

Recall that an object A is said to be *Hopfian* if every epimorphism $A \rightarrow A$ is one-to-one and hence an isomorphism. If A satisfies the ascending chain condition on kernels, then it is easy to see that A is Hopfian. Indeed, if N is maximal with $A/N \cong A$, then A/N can have no proper homomorphic image isomorphic to A , and hence $A/N \cong A$ is Hopfian. As a consequence, we have

COROLLARY 1.5: *If all homomorphic images of A are finitely presented, then A is Hopfian.*

We close this section with three lemmas that offer some converses and analogs to Lemma 1.3 for our specific objects of interest. In all of these situations, we will have $A = \langle B, C \rangle$ so that A is somehow generated by B and C . If B is finitely generated by the generator set β and C is finitely generated by the generator set γ , then surely A is finitely generated by $\beta \cup \gamma$. Furthermore, if B and C are finitely presented, then $\beta \cup \gamma$ satisfies the finitely many relations of β and of γ . Furthermore, depending on the particular structure of A , we must adjoin additional relations that are mentioned in each of the three lemmas. We will use this notation and observation below. We start with groups.

LEMMA 1.6: *Let B and C be multiplicative groups.*

- i. B and C are finitely generated (respectively, finitely presented) if and only if their direct product $A = B \times C$ is finitely generated (respectively, finitely presented).*
- ii. B and C are finitely generated (respectively, finitely presented) if and only if their free product $A = B * C$ is finitely generated (respectively, finitely presented).*

Proof. (i) We know from Lemma 1.3 that if A is finitely generated or finitely presented, then so are B and C . Conversely if B and C are finitely generated, then so is $A = \langle B, C \rangle$. Furthermore, if B and C are finitely presented, then we

must adjoin to the relations for β and γ , the finitely many relations that assert that each element of β commutes with each element of γ .

(ii) If B and C are finitely generated, then so is $A = \langle B, C \rangle$. Furthermore, if B and C are finitely presented, then freeness implies that no additional relations need be adjoined to the finite number of relations for β and for γ . Thus $A = B * C$ is finitely presented.

Conversely, note that we have epimorphisms $\theta_B: A \rightarrow B$ given by $B * C \rightarrow B * 1 \cong B$ and $\theta_C: A \rightarrow C$. Thus if A is finitely generated, then so are B and C . Furthermore, if A is finitely presented, then since the kernel of θ_B is generated by C and hence by the finite set γ , we conclude from Lemma 1.2 that B is finitely presented. Similarly, C is also finitely presented. ■

The proof of the next lemma is identical to that of part (i) above.

LEMMA 1.7: *Let B and C be Lie algebras over the field K . Then B and C are finitely generated (respectively, finitely presented) if and only if their direct sum $A = B \oplus C$ is finitely generated (respectively, finitely presented).*

Finally, we consider associative algebras.

LEMMA 1.8: *Let B and C be associative K -algebras.*

- i. *B and C are finitely generated (respectively, finitely presented) if and only if their direct sum $A = B \oplus C$ is finitely generated (respectively, finitely presented).*
- ii. *B and C are finitely generated (respectively, finitely presented) if and only if their tensor product $A = B \otimes_K C$ is finitely generated (respectively, finitely presented).*

Proof. (i) Again, we know from Lemma 1.3 that if $A = B \oplus C$ is finitely generated or finitely presented then so are B and C . Conversely if B and C are finitely generated, then so is $A = \langle B, C \rangle$. Furthermore, suppose B and C are finitely presented and let 1_B and 1_C be their corresponding identity elements. Then we adjoin to the finite number of relations for β and γ , the relation $1_B 1_C = 1_C 1_B = 0$. In this way, 1_B and 1_C become orthogonal idempotents whose sum clearly acts like the identity on A . Thus $A \cong B \oplus C$ is finitely presented.

(ii) If B and C are finitely generated, then so is $A = \langle B, C \rangle$. Furthermore, if B and C are finitely presented, then we must adjoin to the relations for β and

γ , the finitely many relations that assert that each element of β commutes with each element of γ . In this way, $A = B \otimes C$ is finitely presented.

Conversely, suppose $A = B \otimes C$ is finitely generated. Then A has countable dimension over K , and hence so does B . The latter implies that we can write B as an ascending union $B = \bigcup_m B_m$ of finitely generated subalgebras. Hence A is the ascending union of the various $B_m \otimes C$. But A is finitely generated, so this union must terminate in a finite number of steps. Thus $B \otimes C = A = B_m \otimes C$ for some m and $B = B_m$ is finitely generated. Similarly, C is a finitely generated algebra.

Finally, suppose $A = B \otimes C$ is finitely presented. Then A and B are finitely generated, so we can map a finitely generated free algebra F onto B with kernel I . Then $F \otimes C$ maps onto $B \otimes C = A$ with kernel $I \otimes C$, and by Lemma 1.1, $I \otimes C$ is finitely generated as an ideal. Now I has countable dimension over K , so we can write I as an ascending union $\bigcup_n I_n$ of ideals I_n , each of which is finitely generated as an ideal. Hence $I \otimes C$ is the ascending union of the various $I_n \otimes C$. But $I \otimes C$ is finitely generated as an ideal, so this union necessarily terminates in a finite number of steps. Thus $I \otimes C = I_n \otimes C$ for some n , and hence $I = I_n$ is finitely generated as an ideal of F . It follows that $B \cong F/I$ is finitely presented, and similarly so is C . ■

2. Groups and their Group Rings

Let G be a multiplicative group and let R be a ring. Then the group ring $R[G]$ is clearly an R -algebra as defined in the previous section. Notice that if $N \triangleleft G$, then the epimorphism $G \rightarrow G/N$ extends to an epimorphism $\varphi_N: R[G] \rightarrow R[G/N]$. In particular, when $N = G$ we have an epimorphism $\varphi_G: R[G] \rightarrow R$ sending each group element to 1. This is known as the *augmentation map* and its kernel is the *augmentation ideal* $\omega(R[G])$. Clearly the latter is the set of group ring elements with coefficient sum 0 and hence it is the R -linear span of the elements $1 - g$ with $g \in G$. More generally, the kernel of φ_N is easily seen to be the two-sided ideal $\omega(R[N]) \cdot R[G]$ and this is generated by all $1 - g$ with $g \in N$.

LEMMA 2.1: *N is finitely generated as a normal subgroup of G if and only if $\omega(R[N]) \cdot R[G]$ is finitely generated as an ideal of $R[G]$.*

Proof. First suppose N is generated as a normal subgroup by $g_1, g_2, \dots, g_n \in N$ and let I be the ideal of $R[G]$ generated by the elements $1 - g_i$. Then certainly

$I \subseteq \omega(R[N]) \cdot R[G]$. Conversely, we have a group homomorphism α from $G \subseteq R[G]$ into the group of units of $R[G]/I$ and notice that g_1, g_2, \dots, g_n are in the kernel of this map. Thus since the kernel of α is a normal subgroup of G , we must have $N \subseteq \ker \alpha$. Thus $\omega(R[N]) \subseteq I$ and $I = \omega(R[N]) \cdot R[G]$ is finitely generated by the elements $1 - g_i$.

In the other direction, let $\omega(R[N]) \cdot R[G]$ be finitely generated as an ideal. Since this ideal is generated by the various $1 - g$ with $g \in N$, it follows that it is finitely generated by $1 - g_1, 1 - g_2, \dots, 1 - g_n$ for some $g_i \in N$. Now let M be the normal subgroup of G generated by g_1, g_2, \dots, g_n . Then M is finitely generated as a normal subgroup of G , $M \subseteq N$, and each $1 - g_i$ is contained in $\omega(R[M]) \cdot R[G] \subseteq \omega(R[N]) \cdot R[G]$. But the elements $1 - g_i$ generate the larger ideal, so we must have $\omega(R[M]) \cdot R[G] = \omega(R[N]) \cdot R[G]$, and hence $N = M$ is finitely generated as a normal subgroup. ■

Now we need one particular example of interest. Namely we show that a group ring of a finitely generated free group is finitely presented.

LEMMA 2.2: *Let $G = \langle g_1, g_2, \dots, g_n \rangle$ be the free group on the n free generators g_1, g_2, \dots, g_n , and let R be any ring. Then the group ring $R[G]$ is finitely presented as an R -algebra.*

Proof. Let $F = R\langle x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \rangle$ be the free R -algebra on the $2n$ free generators x_i, y_i that commute with R . Then there exists an R -epimorphism $\pi: F \rightarrow R[G]$ given by $\pi(x_i) = g_i$ and $\pi(y_i) = g_i^{-1}$. The kernel of π clearly contains the $2n$ elements $x_i y_i - 1$ and $y_i x_i - 1$. Now let I be the ideal of F generated by the various elements $x_i y_i - 1$ and $y_i x_i - 1$, and let $\bar{}$ denote the epimorphism $F \rightarrow F/I$. Then π factors through $\bar{}$ so there exists a homomorphism $\bar{\pi}: \bar{F} \rightarrow R[G]$ with $\bar{\pi}(\bar{x}_i) = g_i$ and $\bar{\pi}(\bar{y}_i) = g_i^{-1}$. On the other hand, note that \bar{x}_i is invertible in \bar{F} with inverse \bar{y}_i , so since G is free, there is a homomorphism ϑ from G to the units of \bar{F} with $\vartheta(g_i) = \bar{x}_i$ and $\vartheta(g_i^{-1}) = \bar{y}_i$. Of course, ϑ extends to an epimorphism $\vartheta: R[G] \rightarrow \bar{F}$. Since $\bar{\pi}$ and ϑ are clearly inverses of each other, we see that both are one-to-one and hence $I = \ker \pi$. Thus $\ker \pi$ is finitely generated as an ideal, and $R[G]$ is finitely presented as an R -algebra. ■

The following can be found in Baumslag [Bm] with a proof that is perhaps a bit too skimpy.

THEOREM 2.3: *Let G be a multiplicative group and let R be a ring. Then G is finitely presented as a group if and only if its group ring $R[G]$ is finitely presented as an R -algebra.*

Proof. Suppose first that G is finitely presented. Then there exists a finitely generated free group F and an epimorphism $\pi: F \rightarrow G$ such that $N = \ker \pi$ is finitely generated as a normal subgroup of F . By Lemma 2.1, the corresponding epimorphism $\pi: R[F] \rightarrow R[F/N] \cong R[G]$ has a kernel that is finitely generated as an ideal. Thus since $R[F]$ is finitely presented by the previous lemma, we conclude from Lemma 1.2 that $R[G]$ is finitely presented.

Conversely suppose $R[G]$ is finitely presented as an R -algebra. Then $R[G]$ is finitely generated over R and hence G is finitely generated by the supports of the finite number of generators of $R[G]$. In particular, there exists a finitely generated free group F with $G \cong F/N$ for some normal subgroup N . Since $R[G]$ is finitely presented, the kernel of the epimorphism $R[F] \rightarrow R[F/N] \cong R[G]$ is finitely generated as an ideal by Lemma 1.1 and hence N is finitely generated as a normal subgroup by Lemma 2.1. Thus G is indeed finitely presented. ■

The proof of the next result uses a simplification of a group construction due to Abels [A].

LEMMA 2.4: *Let G be a finitely generated infinite group. Then the wreath product $\mathbb{Z} \wr G$ is a finitely generated group that is not finitely presented.*

Proof. Let $S = \mathbb{Z}[G]$ be the integral group ring of G . We consider the group \mathcal{G} of 3×3 matrices over S of the form

$$[g, a, b, c] = \begin{bmatrix} 1 & a & c \\ & g & b \\ & & 1 \end{bmatrix}$$

with $g \in G$ and $a, b, c \in S$. It can be shown that this group is finitely generated with an infinitely generated center. However we are actually concerned with a more interesting, somewhat smaller subgroup of \mathcal{G} .

To start with, for each $g \in G$, write $\bar{g} = [g, 0, 0, 0]$ and let $\bar{G} = \{\bar{g} \mid g \in G\}$. Then the map $G \rightarrow \bar{G}$ given by $g \mapsto \bar{g}$ is clearly an isomorphism, so \bar{G} is a finitely generated subgroup of \mathcal{G} . Next, note that the map $\mathcal{G} \rightarrow G$ given by $[g, a, b, c] \mapsto g$ is an epimorphism with kernel \mathcal{H} , the set of all matrices of the form $[1, a, b, c]$ with $a, b, c \in S$. Thus \mathcal{H} is a normal subgroup of \mathcal{G} and clearly

$\mathcal{G} = \mathcal{H} \rtimes \bar{G}$, the semidirect product of \mathcal{H} by \bar{G} . Of course, the set of matrices $\mathcal{Z} \subseteq \mathcal{G}$ of the form $[1, 0, 0, c]$ with $c \in S$ is central in \mathcal{G} and is isomorphic to the additive subgroup of S . Thus the center of \mathcal{G} has an infinitely generated torsion-free subgroup and hence is not finitely generated.

Now let $*$ be the classical antiautomorphism of $S = \mathbb{Z}[G]$ determined by $g^* = g^{-1}$ for all $g \in G$, and let \mathcal{H}^* be the subset of \mathcal{H} given by all elements $[1, a, b, c]$ with $b = a^*$. Since we have

$$\text{(mult)} \quad [1, a, a^*, c][1, b, b^*, d] = [1, a + b, a^* + b^*, c + d + ab^*]$$

it follows easily that \mathcal{H}^* is closed under multiplication and inverses. Hence \mathcal{H}^* is a subgroup of \mathcal{H} . Furthermore, $\mathcal{H}^* \supseteq \mathcal{Z}$ and $\mathcal{H}^*/\mathcal{Z}$ is isomorphic to the additive subgroup of S via the map $[1, a, a^*, c] \mapsto a$. Next, for any $g \in G$, we have

$$\text{(conj)} \quad \bar{g}^{-1}[1, a, a^*, c]\bar{g} = [1, ag, g^{-1}a^*, c] = [1, ag, (ag)^*, c]$$

so \bar{G} normalizes \mathcal{H}^* . Thus $\mathcal{G}^* = \mathcal{H}^*\bar{G} \cong \mathcal{H}^* \rtimes G$ is a subgroup of \mathcal{G} .

Finally, let \mathcal{W} be the finitely generated subgroup of \mathcal{G}^* generated by \bar{G} and $[1, 1, 1, 0] \in \mathcal{H}^*$. Since $\bar{g}^{-1}[1, 1, 1, 0]g = [1, g, g^*, 0]$, it follows from equation **(mult)** that for all $a \in S$, there exists some $c \in S$, depending on a , with $[1, a, a^*, c] \in \mathcal{W}$. We can also use equation **(mult)** to compute the commutator of two elements of \mathcal{W} , say $[1, a, a^*, c]$ and $[1, b, b^*, d]$. Specifically, this commutator $[1, 0, 0, r]$ satisfies

$$[1, a, a^*, c][1, b, b^*, d] = [1, b, b^*, d][1, a, a^*, c][1, 0, 0, r]$$

and hence

$$\begin{aligned} [1, a + b, a^* + b^*, c + d + ab^*] &= [1, a + b, a^* + b^*, c + d + ba^*][1, 0, 0, r] \\ &= [1, a + b, a^* + b^*, c + d + r + ba^*]. \end{aligned}$$

Thus $r = ab^* - ba^*$, and by taking $b = 1$, we see that $[1, 0, 0, a - a^*] \in \mathcal{W} \cap \mathcal{Z}$ for all $a \in S$.

Since G is infinite, it follows that $\mathcal{W} \cap \mathcal{Z}$ is an infinitely generated central subgroup of \mathcal{W} . Thus, since \mathcal{W} is finitely generated, Lemma 1.1 implies that the group $\mathcal{W}/(\mathcal{W} \cap \mathcal{Z})$ is finitely generated, but not finitely presented. It remains to understand the latter factor group. To this end, note that $\mathcal{W}\mathcal{Z} = \mathcal{G}^*$, so

$$\mathcal{W}/(\mathcal{W} \cap \mathcal{Z}) \cong \mathcal{G}^*/\mathcal{Z} \cong (\mathcal{H}^*/\mathcal{Z}) \rtimes \bar{G} \cong (\mathcal{H}^*/\mathcal{Z}) \rtimes G.$$

Furthermore, $\mathcal{H}^*/\mathcal{Z}$ is isomorphic to the additive group of S , and G acts on S via right multiplication. Thus $\mathcal{W}/(\mathcal{W} \cap \mathcal{Z}) \cong S \rtimes G$. Now, S additively is the free abelian group with \mathbb{Z} -basis G , and G acts on S by regularly permuting this basis. By definition, this means that $S \rtimes G$ is isomorphic to the wreath product $\mathbb{Z} \wr G$, and therefore the lemma is proved. ■

For example, in the above lemma, we can take $G = \mathbb{Z}$ to be infinite cyclic. Then we conclude that $\mathbb{Z} \wr \mathbb{Z}$ is finitely generated but not finitely presented. This group is of course the semidirect product of the free abelian group A , with free generators $\{a_i \mid i \in \mathbb{Z}\}$, by the infinite cyclic group $\langle x \rangle$, with $x^{-1}a_i x = a_{i+1}$ for all i .

Next, note that there are numerous examples of finitely presented groups that are not Hopfian. Most well-known are the Baumslag-Solitar groups

$$BS(m, n) = \langle a, b \mid a^{-1}b^m a = b^n \rangle$$

where the parameters m and n are of course nonzero integers. Indeed, according to [BS, Theorem 1], $BS(m, n)$ is Hopfian if and only if m or n divides the other, or m and n have precisely the same prime divisors.

In particular, the group

$$G = BS(2, 3) = \langle a, b \mid a^{-1}b^2 a = b^3 \rangle$$

is non-Hopfian. To see this, note that the subgroup of G generated by a and b^2 contains $a^{-1}b^2 a = b^3$ and hence b , so $\langle a, b^2 \rangle = \langle a, b \rangle = G$. Furthermore, by squaring both sides of the relation $a^{-1}b^2 a = b^3$, we get $a^{-1}(b^2)^2 a = (b^2)^3$, and thus there exists an epimorphism $\pi: G \rightarrow G$ given by $\pi(a) = a$ and $\pi(b) = b^2$. Finally, it follows from work of Magnus [M] that $a^{-1}ba$ does not commute with b in G . In particular, the commutator $[a^{-1}ba, b]$ is not 1. But

$$\pi([a^{-1}ba, b]) = [a^{-1}b^2 a, b^2] = [b^3, b^2] = 1,$$

so $[a^{-1}ba, b]$ is a nonidentity element in $\ker \pi$.

We can of course use the group $G = BS(2, 3)$ to obtain a finitely presented, non-Hopfian group ring. Indeed, let R be any ring with 1. Then by Theorem 2.3, $R[G]$ is a finitely presented group ring. Furthermore, the group epimorphism $\pi: G \rightarrow G$ extends to a group ring epimorphism $\pi': R[G] \rightarrow R[G]$. Since $0 \neq 1 - [a^{-1}ba, b]$ is in the kernel of π' , we conclude that $R[G]$ is non-Hopfian.

A group is said to be *coherent* if every finitely generated subgroup is finitely presented. Certainly, every free group is coherent. The following example, due

to Stallings in [Sta], shows that finitely presented groups are not necessarily coherent. We offer a group theoretic proof here rather than the original topological one. For convenience, we write $\mathbf{fg}\langle x, y \rangle$ for the free group on the two generators.

THEOREM 2.5: *Let $T = \mathbf{fg}\langle a, b \rangle \times \mathbf{fg}\langle c, d \rangle$ be the direct product of the two free groups and let S be the three generator subgroup of T given by $S = \langle a, bc, d \rangle$. Then $S \triangleleft T$ with T/S infinite cyclic. Furthermore, T is finitely presented but S is not. In particular, T is not coherent.*

Proof. Since free groups are finitely presented, we know that T is finitely presented by Lemma 1.6(i). We now proceed in a series of three steps, first describing the structure of S and then its relations. Finally we show that these relations are not finitely generated as a normal subgroup. In this way we deduce that S is not finitely presented. ■

STEP 1: *The structure of S .*

Proof. As above, we write $\mathbf{fg}\langle x, y \rangle$ for the free group on the two generators. Then we have an epimorphism $\mathbf{fg}\langle x, y \rangle \rightarrow \langle y \rangle$ given by $x \mapsto 1, y \mapsto y$ and we denote its kernel by $\mathbf{fg}\langle x, y \rangle_x$. This kernel is the normal subgroup of $\mathbf{fg}\langle x, y \rangle$ generated as a subgroup by all conjugates $x^{(y^n)}$ for $n \in \mathbb{Z}$. Indeed, the subgroup of $\mathbf{fg}\langle x, y \rangle$ generated by all y^n -conjugates of x is clearly normalized by x and y , and when one mods out by this subgroup, only y remains. Similarly, we write $\mathbf{fg}\langle x, y \rangle_y$ for the subgroup of $\mathbf{fg}\langle x, y \rangle$ generated by all conjugates $y^{(x^n)}$ with $n \in \mathbb{Z}$.

Now the projection of S onto the first factor of T is given by $a \mapsto a, bc \mapsto b$ and $d \mapsto 1$. Thus $\langle a, bc \rangle$ is free of rank 2, and similarly $\langle bc, d \rangle$ is also free of rank 2. Since c commutes with a and b , we see that $a^{((bc)^n)} = a^{(b^n)}$ and hence $\mathbf{fg}\langle a, bc \rangle_a = \mathbf{fg}\langle a, b \rangle_a \subseteq \mathbf{fg}\langle a, b \rangle$. In other words, the latter two “sub a ” groups are identical as subgroups of T . Note that the first formulation shows that the subgroup is in S and is normalized by a and bc , while the second formulation shows that it is centralized by c and d . Thus this group is normal in T . Similarly we have $\mathbf{fg}\langle bc, d \rangle_d = \mathbf{fg}\langle c, d \rangle_d \subseteq \mathbf{fg}\langle c, d \rangle$ is normal in T . Set $S_0 = \mathbf{fg}\langle a, b \rangle_a \times \mathbf{fg}\langle c, d \rangle_d \subseteq \mathbf{fg}\langle a, b \rangle \times \mathbf{fg}\langle c, d \rangle$ so $S_0 \subseteq S, S_0 \triangleleft T$ and T/S_0 is naturally isomorphic to the free abelian group $\langle b, c \rangle$. Furthermore, S/S_0 corresponds to the subgroup $\langle bc \rangle$, so S/S_0 is infinite cyclic. In addition, $S \triangleleft T$ and T/S is infinite cyclic. ■

STEP 2: *The relations of S.*

Proof. Let $F = \mathbf{fg}\langle x, y, z \rangle$ be the free group on generators x, y, z and consider the epimorphism $\varphi: F \rightarrow S$ given by $\varphi(x) = a$, $\varphi(y) = bc$ and $\varphi(z) = d$. We will precisely determine the kernel of φ . To this end, let N be the normal subgroup of F generated, as a normal subgroup, by the commutators

$$g(m, n) = [x^{(y^m)}, z^{(y^n)}] \quad \text{for all } m, n \in \mathbb{Z}.$$

Since the image under φ of $x^{(y^m)}$ is contained in $\mathbf{fg}\langle a, bc \rangle_a \subseteq \mathbf{fg}\langle a, b \rangle$ and the image of $z^{(y^n)}$ is contained in $\mathbf{fg}\langle bc, d \rangle_d \subseteq \mathbf{fg}\langle c, d \rangle$, it is clear that $N \subseteq \ker \varphi$. We will prove the equality of these two normal subgroups by looking closer at the structure of F/N .

Let $\bar{\cdot}: F \rightarrow F/N$ denote the natural epimorphism. Since $N \subseteq \ker \varphi$, the map φ factors through F/N and there is an epimorphism $\bar{\varphi}: \bar{F} \rightarrow S$ given by $\bar{\varphi}(\bar{x}) = a$, $\bar{\varphi}(\bar{y}) = bc$ and $\bar{\varphi}(\bar{z}) = d$. Note that $\bar{\varphi}$ maps $\langle \bar{x}, \bar{y} \rangle$ onto $\langle a, bc \rangle$ and the latter group is free on the two generators. Thus $\langle \bar{x}, \bar{y} \rangle$ is also free and $\bar{\varphi}: \mathbf{fg}\langle \bar{x}, \bar{y} \rangle \rightarrow \mathbf{fg}\langle a, bc \rangle$ is an isomorphism. In particular, we see that the map $\bar{\varphi}: \mathbf{fg}\langle \bar{x}, \bar{y} \rangle_{\bar{x}} \rightarrow \mathbf{fg}\langle a, bc \rangle_a$ is also an isomorphism. Note that $\mathbf{fg}\langle \bar{x}, \bar{y} \rangle_{\bar{x}}$ is normalized by \bar{x} and \bar{y} . Furthermore it is centralized by \bar{z} since $g(m, 0) \in N$ for all $m \in \mathbb{Z}$. Thus $\mathbf{fg}\langle \bar{x}, \bar{y} \rangle_{\bar{x}}$ is normal in \bar{F} and similarly so is $\mathbf{fg}\langle \bar{y}, \bar{z} \rangle_{\bar{z}}$. Note that the relations $g(m, n) \in N$ imply that these two normal subgroups commute elementwise. Hence $\bar{F}_0 = \mathbf{fg}\langle \bar{x}, \bar{y} \rangle_{\bar{x}} \cdot \mathbf{fg}\langle \bar{y}, \bar{z} \rangle_{\bar{z}}$ is clearly the direct product of the two normal subgroups and we conclude easily that $\bar{\varphi}: \bar{F}_0 \rightarrow S_0$ is an isomorphism.

Finally note that S/S_0 is the infinite cyclic group generated by the image of bc , and \bar{F}/\bar{F}_0 is cyclic, generated by the image of \bar{y} . With this, we conclude that $\bar{\varphi}$ is an isomorphism and hence $N = \ker \varphi$. In other words, N is the normal subgroup of relations of S . ■

STEP 3: *N is not finitely generated as a normal subgroup of F and hence S is not finitely presented.*

Proof. If N is finitely generated as a normal subgroup, then it is generated as a normal subgroup by finitely many of the elements $g(m, n)$. Choose an integer r larger than this number of $g(m, n)$ generators. Then we have less than r remainders $n - m \pmod r$ determined by these generators and so there is at least one remainder, say k , that is missing. In particular, N is generated as a normal

subgroup by all the elements $g(m, n)$ that satisfy $n - m \not\equiv k \pmod r$. The goal is to show that these relations do not imply the remaining ones, and for this we need some sort of wreath product structure.

Consider the symmetric group G on $\{0, 1, \dots, r-1\}$ and with two additional symbols $*$ and \bullet . Let u and w be the two transpositions $u = (*, k)$ and $w = (\bullet, 0)$ and let v be the r -cycle $v = (0, 1, \dots, r-1)$. Since $u^{(v^m)} = (*, m+k \pmod r)$ and $w^{(v^n)} = (\bullet, n \pmod r)$, and since transpositions commute if and only if they are identical or disjoint, we see that $u^{(v^m)}$ commutes with $w^{(v^n)}$ if and only if $m+k \equiv n \pmod r$ or equivalently $n-m \equiv k \pmod r$. In particular, if θ is the homomorphism $\theta: F \rightarrow G$ given by $\theta(x) = u$, $\theta(y) = v$ and $\theta(z) = w$, then $\ker \theta \triangleleft F$ contains $g(m, n)$ if and only if $n-m \equiv k \pmod r$. Thus those $g(m, n)$ in $\ker \theta$ cannot generate all the $g(m, n)$ as a normal subgroup of F . With this contradiction, we conclude from Lemma 1.1 that S is not finitely presented. ■

3. Lie Algebras and their Enveloping Algebras

Let A be an associative K -algebra and define the map $[\ , \]: A \times A \rightarrow A$ by $[a, b] = ab - ba$ for all $a, b \in A$. Then it is easy to verify that $[\ , \]$ is bilinear, skew-symmetric and satisfies the Jacobi identity. Thus, in this way, the elements of A form a Lie algebra which we denote by $\mathfrak{L}(A)$. Note that if $\sigma: A \rightarrow B$ is an algebra homomorphism, then the same map determines a Lie homomorphism $\sigma: \mathfrak{L}(A) \rightarrow \mathfrak{L}(B)$. Of course, if L is an arbitrary Lie algebra, then there is no reason to believe that L is equal to some $\mathfrak{L}(A)$. However it is true that each such L is a Lie subalgebra of some $\mathfrak{L}(A)$ and in some sense, the largest choice of A , with A generated by L , is the universal enveloping algebra $U(L)$.

The construction of $U(L)$ starts with its universal definition. Let L be fixed and consider the set of all pairs (A, θ) , where A is a K -algebra and $\theta: L \rightarrow \mathfrak{L}(A)$ is a Lie homomorphism. As usual, if $\sigma: A \rightarrow B$ is an algebra homomorphism, then the composite map $\sigma\theta: L \rightarrow \mathfrak{L}(B)$ is a Lie homomorphism and hence $(B, \sigma\theta)$ is an allowable pair. A *universal enveloping algebra* for L is therefore defined to be a pair (U, θ) such that, for any other pair (B, ϕ) , there exists a unique algebra homomorphism $\sigma: U \rightarrow B$ with $\phi = \sigma\theta$. It is fairly easy to prove that (U, θ) exists and that it is unique up to suitable isomorphism. Unfortunately, the existence proof does not tell us what U really looks like. In particular, without a good deal of work, it does not settle the question of

whether $\theta: L \rightarrow \mathfrak{L}(U)$ is one-to-one. In fact, θ is one-to-one and this is the important Poincaré-Birkhoff-Witt Theorem (see [J, §V.2] for more details).

If X is any set of elements, let F_X be the free K -algebra on the variables X and let L_X be the Lie subalgebra of $\mathfrak{L}(F_X)$ generated by X . Then the various L_X 's play the role of the free Lie algebras in this theory, and with this, we can speak about finitely presented Lie algebras. For example, suppose \bar{L} is a Lie algebra generated by the set \bar{X} and let $\sigma: X \rightarrow \bar{X}$ be a one-to-one correspondence of sets. Then \bar{X} generates $U(\bar{L})$ as an algebra, and σ extends to an epimorphism $\sigma: F_X \rightarrow U(\bar{L})$. The restriction of σ then yields a Lie epimorphism of L_X onto \bar{L} . It follows easily that $U(L_X) = F_X$ and thus the following result of [Bm], the Lie analog of Theorem 2.3, comes as no surprise.

THEOREM 3.1: *Let L be a Lie algebra over the field K . Then L is finitely presented as a Lie algebra if and only if $U(L)$ is finitely presented as an associative K -algebra.*

Next, we consider the Lie algebra variant of Lemma 2.4 using similar but somewhat easier arguments.

LEMMA 3.2: *Let K be a field, let L be a finite-dimensional K -Lie algebra, and let $U = U(L)$ be its enveloping algebra. Suppose that either $\text{char } K \neq 2$ and $L \neq 0$ or $\text{char } K = 2$ and L is not commutative. Then the Lie algebra $U \rtimes L$ is finitely generated but not finitely presented. Here U is viewed as a commutative Lie algebra and the ad action of $\ell \in L$ on U is given by left multiplication.*

Proof. We consider the ring $S = U_3$ of 3×3 matrices over U and we write $\alpha * \beta = \alpha\beta - \beta\alpha$ for all $\alpha, \beta \in S$. Of course, S becomes a Lie algebra under $*$ and, as is to be expected, we consider a certain Lie subalgebra. Recall that U has an antipode σ , so that σ is an antiautomorphism that maps each element of L to its negative. Now for any $a, b \in U$ and $\ell \in L$ let us write $[\ell, a, b]$ for the 3×3 matrix

$$[\ell, a, b] = \begin{bmatrix} 0 & a^\sigma & b \\ & \ell & a \\ & & 0 \end{bmatrix} \in S$$

and let us also write $\bar{\ell} = [\ell, 0, 0]$. Notice that $\bar{L} = \{\bar{\ell} \mid \ell \in L\}$ is a Lie algebra under $*$, naturally isomorphic to L , and that \bar{L} normalizes the set

$$\mathcal{L} = \{[0, a, b] \mid a, b \in U\}$$

since a simple computation yields the relation

$$(\mathbf{ad}) \quad [\ell, 0, 0] * [0, a, b] = \begin{bmatrix} 0 & -a^\sigma \ell & 0 \\ & 0 & \ell a \\ & & 0 \end{bmatrix}$$

and $(\ell a)^\sigma = a^\sigma \ell^\sigma = -a^\sigma \ell$.

Next, we see that \mathcal{L} is a Lie subalgebra of S since

$$(\mathbf{comm}) \quad [0, a, b] * [0, c, d] = [0, 0, a^\sigma c - c^\sigma a] \in \mathcal{L}.$$

Indeed, the latter element is in $\mathcal{Z} \subseteq \mathcal{L}$ where $\mathcal{Z} = \{[0, 0, e] \mid e \in U\}$ is central in $\mathcal{L} + \bar{L} = \mathcal{L} \rtimes \bar{L}$.

Finally, let \mathcal{W} be the finitely generated Lie subalgebra of $\mathcal{L} \rtimes \bar{L}$ generated by \bar{L} and $[0, 1, 0] \in \mathcal{L}$. Since $U(L)$ is generated by L , as a K -algebra, it follows from equation **(ad)** that \mathcal{W} contains all $[0, a, 0]$ with $a \in U$. Hence, by **(comm)**, $\mathcal{W} \cap \mathcal{Z}$ contains all $[0, 0, a^\sigma c - c^\sigma a]$ with $a, c \in U$. Suppose first that $\text{char } k \neq 2$ and $L \neq 0$, and choose $0 \neq \ell \in L$. Then, by taking $a = 1$ and c any odd power of ℓ , we see that $\mathcal{W} \cap \mathcal{Z}$ contains all $[0, 0, d]$ with d an odd power of ℓ . On the other hand, if $\text{char } K = 2$, then σ fixes all elements of L and hence all powers of elements of L . In particular, by taking $a, \ell \in L$ that do not commute and $c = \ell^n$ for any odd integer n , we see that $d = a^\sigma c - c^\sigma a = a\ell^n - \ell^n a = \ell^{n-1}m$, where $m \in L$ is the nonzero Lie product of a and ℓ .

Hence, in both cases, it follows that $\mathcal{W} \cap \mathcal{Z}$ is a non-finitely generated central ideal of \mathcal{W} and, since \mathcal{W} is finitely generated, Lemma 1.1 now implies that $\mathcal{W}/(\mathcal{W} \cap \mathcal{Z})$ is a finitely generated but not finitely presented Lie algebra. A close look at equations **(ad)** and **(comm)** show that $\mathcal{W}/(\mathcal{W} \cap \mathcal{Z})$ is isomorphic to $U \rtimes L$ where U is viewed as an abelian Lie algebra and L acts on U via left multiplication. ■

The special case where $L = Kx$ is 1-dimensional is of interest. Here $U(L) = K[x]$ is the polynomial ring in x so U has K -basis $a_i = x^i$ for $i = 0, 1, 2, \dots$. Then x acts on U via the derivation $d(a_i) = xa_i = a_{i+1}$. This is an example due to Bahturin [Bh], proved using a result of Bryant and Groves [BG].

Let K be a field. A K -Lie algebra is said to be *coherent* if every finitely generated sub-Lie algebra is finitely presented. The following example, due to Roos in [Ro, page 461], shows that finitely presented Lie algebras are not necessarily coherent. We offer an elementary proof here analogous to that of

our Theorem 2.5 and different from the original homological argument. For convenience, we write $\mathfrak{fl}\langle x, y \rangle$ for the free K -Lie algebra on the two generators.

THEOREM 3.3: *Let $T = \mathfrak{fl}\langle a, b \rangle \oplus \mathfrak{fl}\langle c, d \rangle$ be the direct sum of the two free K -Lie algebras and let R be the three generator sub- K -Lie algebra of T that is given by $R = \langle a, b + c, d \rangle$. Then $R \triangleleft T$ with T/R one dimensional. Furthermore, T is finitely presented, but R is not. In particular, T is not coherent.*

Proof. Since finitely generated free Lie algebras are finitely presented, we know that T is finitely presented. We now proceed in a series of three steps. ■

STEP 1: *The structure of R .*

Proof. As above, we write $\mathfrak{fl}\langle x, y \rangle$ for the free K -Lie algebra on the two generators. Then we have an epimorphism $\mathfrak{fl}\langle x, y \rangle \rightarrow Ky$ given by $x \mapsto 0, y \mapsto y$ and we denote its kernel by $\mathfrak{fl}\langle x, y \rangle_x$. This kernel is the ideal of $\mathfrak{fl}\langle x, y \rangle$ generated as a Lie algebra by all elements $x \cdot (\text{ad } y)^n$ for $n \in \mathbb{Z}^+$, the nonnegative integers. Indeed, the sub-Lie algebra of $\mathfrak{fl}\langle x, y \rangle$ generated by all these $x \cdot (\text{ad } y)^n$ is clearly normalized by x and y , and when one mods out by this sub-Lie algebra, only y remains. Similarly, we write $\mathfrak{fl}\langle x, y \rangle_y$ for the sub-Lie algebra of $\mathfrak{fl}\langle x, y \rangle$ generated by all the elements $y \cdot (\text{ad } x)^n$ with $n \in \mathbb{Z}^+$.

Now the projection of R onto the first factor of T is given by $a \mapsto a, b + c \mapsto b$ and $d \mapsto 0$. Thus $\langle a, b + c \rangle$ is free of rank 2, and similarly $\langle b + c, d \rangle$ is also free of rank 2. Since c commutes with a and b , we see, by induction on n , that $a \cdot (\text{ad } (b + c))^n = a \cdot (\text{ad } b)^n$ and hence $\mathfrak{fl}\langle a, b + c \rangle_a = \mathfrak{fl}\langle a, b \rangle_a \subseteq \mathfrak{fl}\langle a, b \rangle$. In other words, the latter two “sub a ” Lie algebras are identical as subalgebras of T . Note that the first formulation shows that the subalgebra is in R and is normalized by a and $b + c$, while the second formulation shows that it is centralized by c and d . Thus this Lie subalgebra is an ideal in T . Similarly we have $\mathfrak{fl}\langle b + c, d \rangle_d = \mathfrak{fl}\langle c, d \rangle_d \subseteq \mathfrak{fl}\langle c, d \rangle$ is also an ideal of T . Set $R_0 = \mathfrak{fl}\langle a, b \rangle_a + \mathfrak{fl}\langle c, d \rangle_d \subseteq \mathfrak{fl}\langle a, b \rangle \oplus \mathfrak{fl}\langle c, d \rangle$ so $R_0 \subseteq R, R_0 \triangleleft T$ and T/R_0 is naturally isomorphic to the two dimensional commutative algebra $Kb + Kc$. Furthermore, R/R_0 corresponds to the subspace $K(b + c)$, so R/R_0 is one dimensional. In addition, $R \triangleleft T$ and T/R is one dimensional. ■

STEP 2: *The relations of R .*

Proof. Let $F = \mathfrak{fl}\langle x, y, z \rangle$ be the free Lie algebra on generators x, y, z and consider the epimorphism $\varphi: F \rightarrow R$ given by $\varphi(x) = a, \varphi(y) = b + c$ and $\varphi(z) = d$.

We will precisely determine the kernel of φ . To this end, let I be the ideal of F generated, as a Lie ideal, by the commutators

$$h(m, n) = [x \cdot (\text{ad } y)^m, z \cdot (\text{ad } y)^n] \quad \text{for all } m, n \in \mathbb{Z}^+.$$

Since the image under φ of $x \cdot (\text{ad } y)^m$ is contained in $\mathfrak{fl}\langle a, b + c \rangle_a \subseteq \mathfrak{fl}\langle a, b \rangle$ and the image of $z \cdot (\text{ad } y)^n$ is contained in $\mathfrak{fl}\langle b + c, d \rangle_d \subseteq \mathfrak{fl}\langle c, d \rangle$, it is clear that $I \subseteq \ker \varphi$. We will prove the equality of these two ideals by looking closer at the structure of the Lie algebra F/I .

Let $\bar{\cdot} : F \rightarrow F/I$ denote the natural epimorphism. Since $I \subseteq \ker \varphi$, the map φ factors through F/I and there is an epimorphism $\bar{\varphi} : \bar{F} \rightarrow R$ given by $\bar{\varphi}(\bar{x}) = a$, $\bar{\varphi}(\bar{y}) = b + c$ and $\bar{\varphi}(\bar{z}) = d$. Note that $\bar{\varphi}$ maps $\langle \bar{x}, \bar{y} \rangle$ onto $\langle a, b + c \rangle$ and the latter Lie algebra is free on the two generators. Thus $\langle \bar{x}, \bar{y} \rangle$ is also free and $\bar{\varphi} : \mathfrak{fl}\langle \bar{x}, \bar{y} \rangle \rightarrow \mathfrak{fl}\langle a, b + c \rangle$ is an isomorphism. In particular, $\bar{\varphi} : \mathfrak{fl}\langle \bar{x}, \bar{y} \rangle_{\bar{x}} \rightarrow \mathfrak{fl}\langle a, b + c \rangle_a$ is also an isomorphism. Note that $\mathfrak{fl}\langle \bar{x}, \bar{y} \rangle_{\bar{x}}$ is normalized by \bar{x} and \bar{y} . Furthermore it is centralized by \bar{z} since $h(m, 0) \in I$ for all $m \in \mathbb{Z}^+$. Thus $\mathfrak{fl}\langle \bar{x}, \bar{y} \rangle_{\bar{x}}$ is an ideal in \bar{F} and similarly so is $\mathfrak{fl}\langle \bar{y}, \bar{z} \rangle_{\bar{z}}$. Note that the relations $h(m, n) \in I$ imply that these two ideals commute elementwise. Hence $\bar{F}_0 = \mathfrak{fl}\langle \bar{x}, \bar{y} \rangle_{\bar{x}} + \mathfrak{fl}\langle \bar{y}, \bar{z} \rangle_{\bar{z}}$ is the direct sum of the two ideals and we conclude easily that $\bar{\varphi} : \bar{F}_0 \rightarrow R_0$ is an isomorphism.

Finally note that R/R_0 is one dimensional generated by the image of $b + c$, and \bar{F}/\bar{F}_0 is generated by the image of \bar{y} . With this, we conclude that $\bar{\varphi}$ is an isomorphism and hence $I = \ker \varphi$. Thus, I is the ideal of relations of R . ■

STEP 3: *I is not finitely generated as a Lie ideal of F and hence R is not finitely presented.*

Proof. If I is finitely generated as an ideal of F , then it is generated as a Lie ideal by finitely many of the elements $h(m, n)$. Choose an integer s larger than the sums $m + n$ for these finitely many generators of I and let M denote the ring of $(s + 3) \times (s + 3)$ matrices over K . We label the rows and columns of M by the numbers $\{0, 1, \dots, s\}$ and the additional symbols $*$ and \bullet . In addition, to avoid subscripts, we let $e(i, j)$ denote the usual matrix units. Now set $u = e(*, 0)$, $w = e(s, \bullet)$ and $v = \sum_{i=0}^{s-1} e(i, i + 1)$.

Working in the Lie algebra $\mathcal{L}(M)$ we see that $u \cdot (\text{ad } v)^m = e(*, m)$ and $w \cdot (\text{ad } v)^n = \pm e(s - n, \bullet)$ for $0 \leq m, n \leq s$. In particular, if θ is the Lie homomorphism $\theta : F \rightarrow \mathcal{L}(M)$ given by $\theta(x) = u$, $\theta(y) = v$ and $\theta(z) = w$, then $\ker \theta \triangleleft F$ contains those $h(m, n)$ with $m + n < s$ since $e(*, i)$ commutes with

$e(j, \bullet)$ if and only if $i \neq j$. But note that $\ker \theta$ does not contain $h(0, s)$, so those $h(m, n)$ with $m + n < s$ cannot generate all $h(m, n)$ as an ideal of F . With this contradiction, we conclude from Lemma 1.1 that R is not finitely presented. ■

It follows from the above and Theorem 3.1 that the enveloping algebra of T is not coherent. In particular, the tensor product of two free algebras of rank 2 is not coherent.

4. Rings and Algebras

Suppose $R \supseteq S$ are rings with the same 1. If R is a finitely generated right or left S -module and if S is finitely generated as a ring, then R is certainly also finitely generated as a ring. Thus it is natural to ask whether the finitely presented property lifts from S to R . As we see below, the answer is “no”.

LEMMA 4.1: *There exist rings $R \supseteq S$ and a central idempotent e of R such that $R = Se + S(1 - e)$. Furthermore, S is finitely presented, but R is not.*

Proof. Let T be a finitely presented ring with a homomorphic image \bar{T} that is not finitely presented. For example, \bar{T} could be any finitely generated ring that is not finitely presented and T could be a finitely generated free ring that maps onto \bar{T} . Now set $R = T \oplus \bar{T}$, so that R is not finitely presented by Lemma 1.3. Furthermore, note that $e = 1 \oplus 0$ is a central idempotent in R . Now embed T into R via the map $t \mapsto t \oplus \bar{t}$ and let S denote the image of T . Then $R = Se + S(1 - e)$ and $S \cong T$ is finitely presented. ■

If R is a right or left Noetherian ring, then all two-sided ideals of R are certainly finitely generated. Thus Lemma 1.2 implies

LEMMA 4.2: *Let R be a right or left Noetherian ring that is finitely presented. Then all homomorphic images of R are also finitely presented.*

As a consequence, we have the well-known fact that every finitely generated commutative ring is finitely presented. In view of this, it is natural to ask whether every affine (that is, finitely generated) PI-algebra is necessarily finitely presented. Using a slightly simpler variant of an example constructed in [SW], but with the same proof, we show below that the answer is “no”.

LEMMA 4.3: *Let F be a field. There exists a finitely generated prime F -algebra, satisfying the multilinear identities of 2×2 matrices, that is not finitely presented.*

Proof. Following [SW], let $A = F[x, y, z]$ be the polynomial ring over F in the three commuting indeterminates. Furthermore, let $I = yA + zA$ and let $R \subseteq M_2(A)$ be given by

$$R = \begin{bmatrix} F + I^2 & I \\ I & A \end{bmatrix}.$$

Then R satisfies the identities of $M_2(A)$ and it is easy to see that R is a finitely generated F -algebra. Furthermore, if P is a nonzero principal prime ideal of A contained in I^2 , then

$$Q = \begin{bmatrix} P & P \\ P & P \end{bmatrix}$$

is a prime ideal of R that is not finitely generated as a 2-sided ideal. Thus Lemma 1.1 implies that the ring R/Q is a finitely generated prime PI-algebra that is not finitely presented. ■

The K -algebras A and B are said to be *Morita equivalent* if their categories of right modules are equivalent. It is known that this occurs if and only if $B \cong eM_n(A)e$ where e is an idempotent of the matrix ring $M_n(A)$ satisfying

$$M_n(A)eM_n(A) = M_n(A).$$

See [L, Proposition 18.33]. Such idempotents e are said to be *full*. According to [MS], the property of being a finitely generated K -algebra is a Morita invariant, namely it carries over from an algebra to a Morita equivalent one. As a generalization, we have the result of [AA2], which asserts that the property of being finitely presented is also a Morita invariant. The proof of this result is clever but simple, so we sketch it below. We start with [AA1, Theorem 1].

PROPOSITION 4.4: *If $A = A_0 \oplus A_1$ is a \mathbb{Z}_2 -graded K -algebra that is finitely presented and if $A_1^2 = A_0$, then the K -subalgebra A_0 is also finitely presented.*

Proof. Since $A_1^2 = A_0$, it follows that A is finitely generated as an algebra with generators in A_1 . Thus we can construct a K -algebra epimorphism θ from the finitely generated free algebra F in the variables $X = \{x_1, x_2, \dots, x_n\}$ onto A with each generator mapping to A_1 . Note that F is also \mathbb{Z}_2 -graded with F_0 spanned by all monomials of even length and F_1 spanned by all monomials

of odd length. Clearly θ preserves the grading, so $\theta(F_0) = A_0$. Furthermore, F_0 is a finitely generated free algebra, with variables $x_i x_j$, so the restriction $\theta_0: F_0 \rightarrow A_0$ is a free presentation of A_0 . It remains to show that $I_0 = \ker \theta_0$ is a finitely generated ideal of F_0 .

Since A is finitely presented and θ preserves the grading, it is clear that $I = \ker \theta$ is generated by a finite set $M_0 \cup M_1$ of homogeneous elements. Then we have

$$I = (F_0 \oplus F_1)(M_0 \cup M_1)(F_0 \oplus F_1)$$

and $F_1 = F_0 X = X F_0$, so we conclude that $I_0 = I \cap F_0$ is generated by the finite set $M_0 \cup M_1 X \cup X M_1 \cup X M_0 X$, as required. ■

Now for the promised result [AA1, Theorem 2] and [AA2, Theorem 1].

THEOREM 4.5: *The property of being a finitely presented K -algebra is a Morita invariant.*

Proof. If A is a finitely presented algebra, then so is any matrix ring $M_n(A)$. Thus we need only show that if A is finitely presented and e is a full idempotent of A , then eAe is finitely presented.

Suppose first that both e and $1 - e$ are full. Then A is \mathbb{Z}_2 -graded with $A_0 = eAe \oplus (1 - e)A(1 - e)$ and $A_1 = eA(1 - e) \oplus (1 - e)Ae$. Furthermore, since e and $1 - e$ are full, it follows that $A_1^2 = A_0$. Thus the previous proposition implies that A_0 is finitely presented and hence so is its direct summand eAe .

Finally, suppose only that e is full and set $\bar{A} = M_2(A)$ and $\bar{e} = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \in \bar{A}$.

Then \bar{A} is finitely presented and \bar{e} is a full idempotent of \bar{A} with $1 - \bar{e}$ also full. Thus by the above,

$$\bar{e}\bar{A}\bar{e} = \begin{bmatrix} eAe & 0 \\ 0 & 0 \end{bmatrix} \cong eAe$$

is indeed finitely presented. ■

Now let V be a right A -module. Then V is said to be *finitely presented* if $V \cong F/U$ where F is a finitely generated free A -module and U is a finitely generated submodule of F . The next result follows easily from Schanuel’s Lemma (see [K, Theorem 1, page 161]).

LEMMA 4.6: *Let V be a finitely presented right A -module, let F' be a finitely generated free A -module and let $\phi: F' \rightarrow V$ be an epimorphism. Then $U' =$*

$\ker \phi$ is a finitely generated submodule of F' and hence $V \cong F'/U'$ is also a finite presentation for the module V .

Proof. Since $0 \rightarrow U \rightarrow F \rightarrow V$ and $0 \rightarrow U' \rightarrow F' \rightarrow V$ are exact, Schanuel's Lemma implies that $F \oplus U' \cong F' \oplus U$. Indeed, this holds even if F and F' are merely assumed to be projective. In particular, U' is a homomorphic image of the finitely generated module $F' \oplus U$. ■

Of course, if A is right Noetherian, then every finitely generated right A -module is finitely presented. For convenience, we introduce a concrete realization of these modules. Specifically, suppose c_1, c_2, \dots, c_n generate the A -module V and let $F = A^n = (A, A, \dots, A)$ be a free A -module of rank n . Then we have an epimorphism $\phi: F \rightarrow V$ given by $\phi(a_1, a_2, \dots, a_n) = \sum_i c_i a_i$. Clearly,

$$U = \ker \phi = \{(a_1, a_2, \dots, a_n) \mid \sum_i c_i a_i = 0\}$$

and we will think of this as the relation module for V corresponding to the generators c_1, c_2, \dots, c_n .

THEOREM 4.7: *Let A be a finitely presented K -algebra and let I be a right ideal of A that is finitely presented as a right A -module. If $AI = A$, then $K + I$ is a finitely presented K -algebra.*

Proof. If $I = A$, the result is trivial, so we can assume that I is proper. Note that $AI = A$ implies that $I \neq 0$. Thus by assumption I has $n \geq 1$ generators c_1, c_2, \dots, c_n with relation module generated by the finitely many relations $g_j = (a_{1j}, a_{2j}, \dots, a_{nj})$. We fix this notation throughout.

Now we look inside the 2×2 matrix ring over A and consider the subalgebra

$$M = \begin{bmatrix} K + I & I \\ A & A \end{bmatrix}.$$

In some sense our goal is to show that M is finitely presented and we proceed in a series of steps. ■

STEP 1: *The construction of B , a finitely presented algebra that mirrors M .*

Proof. First B is a K -algebra with 1, and we need the idempotents corresponding to e_{11} and e_{22} . So B has generators e and f with relations that assert that e and f are orthogonal idempotents that sum to 1.

Next we adjoin the lower right corner of B corresponding to the idempotent f . Specifically, we construct a monomorphism $\bar{\cdot} : A \rightarrow fBf$ with $\bar{1} = f$. Clearly \bar{A} is the K -subalgebra of B generated by the images \bar{a} of the finitely many generators $a \in A$ and subject to their finitely many relations. Furthermore, we insist that for each generator $a \in A$, its image \bar{a} belongs to fBf and this is achieved by adding the relations $f\bar{a} = \bar{a} = \bar{a}f$.

We need just a few more generators and relations. First $r \in B$ plays the role of e_{21} , so we have $fr = r = re$. Further, for $i = 1, 2, \dots, n$, $s_i \in B$ plays the role of $c_i e_{12}$, so $es_i = s_i = s_i f$ and $rs_i = \bar{c}_i \in \bar{A}$, where in the latter relations, the elements \bar{c}_i are written explicitly in terms of the generators of \bar{A} . Finally, for each $g_j = (a_{1j}, a_{2j}, \dots, a_{nj})$, we adjoin the relation

$$\tilde{g}_j = s_1 \bar{a}_{1j} + s_2 \bar{a}_{2j} + \dots + s_n \bar{a}_{nj} = 0$$

where again, each \bar{a}_{ij} is written explicitly in terms of the generators of \bar{A} .

Thus B , as defined above is a finitely presented K -algebra and we have an algebra homomorphism $\theta : B \rightarrow M$ given by $e \mapsto e_{11}$, $f \mapsto e_{22}$, $\bar{a} \mapsto ae_{22}$ for all $a \in A$, $r \mapsto e_{21}$ and $s_i \mapsto c_i e_{12}$ for $i = 1, 2, \dots, n$. Note that, for each j ,

$$\begin{aligned} \theta(\tilde{g}_j) &= c_1 e_{12} a_{1j} e_{22} + c_2 e_{12} a_{2j} e_{22} + \dots + c_n e_{12} a_{nj} e_{22} \\ &= (c_1 a_{1j} + c_2 a_{2j} + \dots + c_n a_{nj}) e_{12} = 0, \end{aligned}$$

as required. We will show below that θ is an isomorphism. ■

STEP 2: B is a direct sum of its four corners

$$B = eBe \oplus eBf \oplus fBe \oplus fBf$$

with $fBf = \bar{A}$, $fBe = \bar{A}r$, $eBf = \sum_i s_i \bar{A}$ and $eBe = Ke + \sum_i s_i \bar{A}r$.

Proof. The first formula follows from

$$B = 1B1 = (e + f)B(e + f)$$

and the four summands here are called the corners of B . Notice that a product of corners is either 0 or contained in another corner. In particular, since all generators of B (other than 1) are contained in corners, we can determine the corners of B by considering which products of generators they contain.

We remark that in the argument below, each product π' is either empty and hence equal to 1 or necessarily in a corner that is easy to describe from context.

We start with fBf and, by construction, we know that $fBf \supseteq \bar{A}$. Now suppose that $\pi \in fBf$ is a nonzero product of generators. We show by induction on the number of factors in π that $\pi \in \bar{A}$. Now $f\pi = \pi \neq 0$ so the left-most factor of π is f , r or \bar{a} for some generator $a \in A$. In the first and third cases we have $\pi = f\pi'$ or $\bar{a}\pi'$ where $\pi' \in fBf$ is a shorter product. By induction, $\pi' \in \bar{A}$ and then $\pi \in \bar{A}$. On the other hand, if π starts with r , then this factor can only be followed by several e 's and then some s_i . But then $re^t s_i = \bar{c}_i$, so $\pi = \bar{c}_i \pi'$ and again $\pi \in \bar{A}$. We conclude that $fBf = \bar{A}$.

Next note that $eBf \supseteq \sum_i s_i \bar{A}$. Conversely suppose $\pi \in eBf$ is a nonzero product of generators. Then $e\pi = \pi \neq 0$ so π starts on the left with e or some s_i . In the former case, $\pi = e\pi'$ and π' is shorter. Thus, by induction, $\pi' \in \sum_i s_i \bar{A}$ and hence $\pi = e\pi' \in \sum_i s_i \bar{A}$. On the other hand, if $\pi = s_i \pi'$, then $\pi' \in fBf = \bar{A}$ and hence $\pi \in s_i \bar{A}$.

The argument for fBe is similar except that we look at the right-most factor of π . Indeed, $fBe \supseteq \bar{A}r$ and if $\pi \in fBe$ is a nonzero product of generators, then $\pi e = \pi \neq 0$, so the right-most factor of π is either e or r . Thus either $\pi = \pi'e$ with π' shorter or $\pi = \pi'r$ with $\pi' \in fBf = \bar{A}$.

Finally, eBe contains Ke and also

$$eBf \cdot fBe = \sum_i s_i \bar{A} \cdot \bar{A}r = \sum_i s_i \bar{A}r.$$

Conversely suppose $\pi \in eBe$ is a nonzero product of generators. If π has just one factor then it is e . Otherwise, $e\pi e = \pi \neq 0$ and ignoring initial and final factors of e , we see that $\pi = s_i \pi' r$ for some i . Then $\pi' \in fBf = \bar{A}$, and hence we have $\pi \in s_i \bar{A}r$. ■

STEP 3: $\theta: B \rightarrow M$ is an isomorphism. Hence M is finitely presented and therefore so is $K + I$.

Proof. Notice that $\theta(\bar{a}) = ae_{22}$ for all $a \in A$, so θ is one-to-one and onto from the f, f corner $fBf = \bar{A}$ of B to the e_{22}, e_{22} corner Ae_{22} of M . Similarly, $\theta(\bar{a}r) = ae_{21}$ shows that θ is one-to-one and onto from the f, e corner $fBe = \bar{A}r$ of B to the e_{22}, e_{11} corner Ae_{21} of M .

Next note that

$$\theta(eBf) = \theta\left(\sum_i s_i \bar{A}\right) = e_{12} \sum_i c_i A = e_{12}I$$

so in this case θ is onto. To check that it is one-to-one on eBf , suppose $\sum_i s_i \bar{a}_i$ maps to 0 for suitable $a_i \in A$. Then

$$0 = \theta\left(\sum_i s_i \bar{a}_i\right) = e_{12} \sum_i c_i a_i$$

so $\sum_i c_i a_i = 0$. Thus (a_1, a_2, \dots, a_n) is in the relation module for c_1, c_2, \dots, c_n and hence it is an A -linear combination of the generators g_j 's. Since

$$\tilde{g}_j = s_1 \bar{a}_{1j} + s_2 \bar{a}_{2j} + \dots + s_n \bar{a}_{nj} = 0$$

in B , by assumption, it therefore follows that $\sum_i s_i \bar{a}_i = 0$.

Finally, note that

$$\theta(eBe) = Ke_{11} + \sum_i e_{12} c_i A e_{22} e_{21} = (K + \sum_i c_i A) e_{11} = (K + I) e_{11}$$

so θ maps eBe onto the e_{11}, e_{11} corner of M . If $\beta = ke + \sum_i s_i \bar{a}_i r$ maps to 0 for some $k \in K$ and $a_i \in A$, then

$$0 = \theta(\beta) = \theta\left(ke + \sum_i s_i \bar{a}_i r\right) = (k + \sum_i c_i a_i) e_{11}$$

and hence $\sum_i c_i a_i = -k$. Since $I \neq A$ by our assumption, it follows that $k = 0$ and then $\sum_i c_i a_i = 0$. As above, we conclude that $\sum_i c_i \bar{a}_i = 0$, so $\sum_i c_i \bar{a}_i r = 0$. We conclude that $\beta = 0$, and θ is indeed one-to-one on eBe .

Thus θ is an isomorphism, so M is isomorphic to B and hence it is finitely presented. Furthermore, since $AI = A$, we have

$$\begin{aligned} Me_{11}M &= \begin{bmatrix} K + I & 0 \\ A & 0 \end{bmatrix} \begin{bmatrix} K + I & I \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} K + I & I \\ A & A \end{bmatrix} = M \end{aligned}$$

and hence $K + I \cong e_{11}Me_{11}$ is Morita equivalent to M . Theorem 4.5 now implies that $K + I$ is also finitely presented. ■

We now consider two examples of interest. First let A be the Weyl algebra over the field K , so that A is generated by x and y subject to the relation $[x, y] = xy - yx = 1$. Then A is the Ore extension $K[x][y; d]$, where d is the ordinary derivation on the polynomial ring $K[x]$. It follows from this formulation that A satisfies: (1) it is finitely presented, (2) it is right and left Noetherian, (3) it is a domain, and (4) y is not a unit in the ring. We actually need a slightly larger

ring, namely A' , the localization of A at the powers of x . Thus A' is the Ore extension $K[x, x^{-1}][y; d]$ and again A' satisfies the above four properties.

Now let $I' = yA'$ be the right ideal of A' generated by y . Then I' is proper since y is not a unit, and it is a free right module of rank 1 since A' is a domain. Furthermore, $A'I' = A'yA'$ is an ideal containing xy and yx and hence 1. Thus $A'I' = A'$ and the preceding theorem implies that the subalgebra $K + I' = K + yA'$ is finitely presented. As we see below, this is not the answer we had hoped for.

Indeed, let W be the Witt Lie algebra so that W has the standard K -basis consisting of the elements e_i , with $i \in \mathbb{Z}$, subject to the commutation relations $[e_i, e_j] = (i-j)e_{i+j}$, for all $i, j \in \mathbb{Z}$. If K has characteristic 0, then it is clear that W is finitely generated. But surprisingly, by [Ste, page 506], W is also finitely presented. Thus, by Theorem 3.1, its universal enveloping algebra $B = U(W)$ is a finitely presented algebra. This algebra has been extensively studied (see for example [DS] and [ISi]), and in particular it was shown in [SiW1] that B is neither right nor left Noetherian. We had hoped to use Lemma 1.2 to exhibit a specific two-sided ideal of $B = U(W)$ that is not finitely generated.

To this end, observe that we have a homomorphism θ from B to A' given by $e_i \mapsto yx^{i+1}$ for all $i \in \mathbb{Z}$, and the image of θ is clearly $K + yA' = K + I'$. If this image were not finitely presented, then by Lemma 1.2, the kernel of θ would be a two-sided ideal of B that is not finitely generated. Alas, as we have seen, $K + I'$ is finitely presented, and hence it follows from Lemma 1.1 that $\ker \theta$ is a finitely generated two-sided ideal.

This argument has been generalized in [SiW2].

5. Hopf Algebras

The group ring and enveloping algebras examples of Lemmas 2.4 and 3.1 surely extend to Hopf algebras. Let H be a Hopf algebra with counit $\varepsilon: H \rightarrow K$, antipode $S: H \rightarrow H$ and comultiplication $\Delta: H \rightarrow H \otimes H$. We consider the ring of 3×3 matrices over H and the subring \mathcal{R} consisting of all elements of the form

$$[h, a, b, c] = \begin{bmatrix} \varepsilon(h) & a & c \\ & h & b \\ & & \varepsilon(h) \end{bmatrix}$$

with $h, a, b, c \in H$. For all $h \in H$, set $\bar{h} = [h, 0, 0, 0]$ and note that the map $h \rightarrow \bar{h}$ defines an isomorphism of Hopf algebras. Conversely, the map $\mathcal{R} \rightarrow H$ given by $[h, a, b, c] \rightarrow h$ is an algebra epimorphism with kernel \mathcal{K} consisting of all elements of \mathcal{R} of the form $[0, a, b, c]$. We have the following product formulas

$$\bar{h} \cdot [0, a, b, c] = [0, \varepsilon(h)a, hb, \varepsilon(h)c]$$

and

$$[0, a, b, c] \cdot \bar{h} = [0, ah, b\varepsilon(h), c\varepsilon(h)].$$

Now we extend the inner action of H on H to an action on \mathcal{K} by defining

$$h * [0, a, b, c] = \sum_{(h)} \bar{h}_1 \cdot [0, a, b, c] \cdot S(\bar{h}_2) = [0, a', b', c']$$

where

$$\begin{aligned} a' &= a \cdot \sum_{(h)} \varepsilon(h_1) S(h_2) = a \cdot S(h), \\ b' &= \sum_{(h)} h_1 \varepsilon(S(h_2)) \cdot b = h \cdot b, \text{ and} \\ c' &= \sum_{(h)} \varepsilon(h_1) \varepsilon(S(h_2)) \cdot c = \varepsilon(h) \cdot c. \end{aligned}$$

In particular, H stabilizes

$$\mathcal{K}_0 = \{[0, a, b, c] \mid a, b, c \in H, a = S(b)\}.$$

6. The Fundamental Problem

The study of finitely presented algebras is an isolated subject. One proves facts about such algebras and also decides whether certain examples are finitely presented or not. The fundamental problem is to use such results in other fields, namely to obtain theorems that do not have “finitely presented” in their statement, but use the concept intrinsically in their proof. At present, we know of only one such result, an application to the study of twisted group algebras. We briefly outline the argument below.

Let $K^t[G]$ denote a twisted group algebra of the multiplicative group G over the field K . Then $K^t[G]$ is an associative K -algebra with K -basis $\bar{G} = \{\bar{x} \mid x \in G\}$ and with multiplication defined distributively by $\bar{x} \cdot \bar{y} = \tau(x, y) \overline{xy}$ for all $x, y \in G$. Here $\tau: G \times G \rightarrow K^\bullet$ is the *twisting function* and, as is well-known,

the associative law in $K^t[G]$ is equivalent to the fact that τ is a 2-cocycle. Furthermore, we can always assume that $\bar{1} = 1$ is the identity element of $K^t[G]$. With this assumption, the *trace map* $\text{tr}: K^t[G] \rightarrow K$ is defined linearly by $\text{tr } \bar{x} = 0$ if $1 \neq x \in G$ and $\text{tr } 1 = 1$. In other words, $\text{tr } \alpha$ is the identity coefficient of $\alpha \in K^t[G]$. It is clear that $\text{tr } \alpha\beta = \text{tr } \beta\alpha$ for all $\alpha, \beta \in K^t[G]$.

Of course, the ordinary group algebra $K[G]$ corresponds to the twisting function with constant value 1. In [K], Kaplansky proved that if e is an idempotent in the complex group algebra $\mathbb{C}[G]$, then $\text{tr } e$ is real and bounded between 0 and 1. Next he observed that if σ is any field automorphism of \mathbb{C} , then σ yields a ring automorphism of $\mathbb{C}[G]$, so $\sigma(e)$ is also an idempotent. Thus $\sigma(\text{tr } e) = \text{tr}(\sigma(e))$ is also real and it follows that $\text{tr } e$ is a totally real algebraic number. This led Kaplansky to conjecture that these traces are actually always rational.

In [Z], Zalesski proved the conjecture via a beautiful two step argument. First, he worked over fields F of characteristic $p > 0$ and used properties of the p -power map to show that $\text{tr } e \in \text{GF}(p)$. Then he considered fields K of characteristic 0, and he constructed suitable places to fields F of characteristic $p > 0$ for infinitely many different primes p . In this way he obtained homomorphisms from subrings of $K[G]$ to $F[G]$. Since $\text{tr } e$ always mapped to elements of the prime subfield of F , the Frobenius Density Theorem allowed him to conclude that $\text{tr } e \in \mathbb{Q}$.

More recently, the paper [P] considered whether this result also holds for twisted group algebras. In characteristic $p > 0$, Zalesski's argument carries over almost verbatim and again one concludes that $\text{tr } e \in \text{GF}(p)$. The difficulty occurs in fields of characteristic 0, when trying to map subrings of $K^t[G]$ to $F^t[G]$. Indeed, note that the basis \bar{G} generates a subgroup \mathcal{G} of the unit group of $K^t[G]$ and certainly we expect $\mathcal{G} \cap K^\bullet$ to be nontrivial. Thus, when we map a subring R of $K^t[G]$ to $F^t[G]$, we need $R \supseteq \mathcal{G}$, but we also need $\mathcal{G} \cap K^\bullet$ to not map to 0. Unfortunately, it is easy to construct examples, using Lemma 2.4, of twisted group algebras where $\mathcal{G} \cap K^\bullet$ fills up the entire nonzero part of the field. Thus the best we can do with this part of Zalesski's argument is to prove the result under the assumption that $\mathcal{G} \cap K^\bullet$ is a finitely generated group.

So where do we go from here? Write $e = \sum_{i=1}^n k_i \bar{x}_i \in K^t[G]$. Then certainly we can assume that $G = \langle x_1, x_2, \dots, x_n \rangle$ is finitely generated. Furthermore, the equality $e^2 = e$ is equivalent to a finite number of group relations of the form $x_r x_s = x_t$ and $x_r x_s = x_u x_v$, along with formulas relating the coefficients and twisting. With a little care, (see [P, Lemma 3]), one can then show that there is

a finitely presented group H , a twisted group algebra $K^t[H]$ and an idempotent $f \in K^t[H]$ such that $\text{tr } e = \text{tr } f$. This allows us to assume that the group G above is also finitely presented.

Now by changing the basis by factors of K^\bullet if necessary, we can clearly assume that \mathcal{G} is generated by $\overline{x_1}, \overline{x_2}, \dots, \overline{x_n}$. Furthermore, note that the map $k\overline{x} \mapsto x$ yields a homomorphism from \mathcal{G} onto G with kernel $\mathcal{G} \cap K^\bullet$. Thus since \mathcal{G} is finitely generated and G is finitely presented, Lemma 1.1 implies that $\mathcal{G} \cap K^\bullet$ is finitely generated as a normal subgroup of \mathcal{G} . Thus this central subgroup of \mathcal{G} is finitely generated as a group and hence we can construct numerous maps from $K^t[G]$ to twisted group algebras in fields of positive characteristic where, as we said, Zaleski's trace argument applies directly. This completes the proof of [P, Theorem 5(i)], a result that does not mention finitely presented groups, but certainly uses them.

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