Groups with All Irreducible Modules of Finite Degree

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Abstract. Let $K[G]$ denote the group algebra of the multiplicative group G over a field K of characteristic 0. As is well known, G has an abelian subgroup of finite index if and only if all irreducible modules of $K[G]$ have finite bounded degree. An open question of interest is whether G must have an abelian subgroup of finite index if $K[G]$ has all irreducible modules of finite degree, but without assuming a bound on these degrees. In this note, we discuss the unpublished thesis work of the second author which comes close to yielding an affirmative solution to this problem.

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In May 1998, I attended the International Algebra Conference at Moscow State University dedicated to the memory of Prof. A. G. Kurosh. My talk there concerned some recent work on the semiprimitivity problem for group rings of locally finite groups over fields of characteristic $p > 0$. Since that material has been adequately surveyed in [4] and [5], it seemed more appropriate to use this opportunity to discuss another group ring problem and specifically to highlight, streamline, and slightly extend the unpublished thesis work of my student Will Temple [6]. I am delighted that Will has agreed to be a coauthor of this paper since, frankly, the results are his. My contribution here is mostly concerned with the writing of the paper and the form of the presentation. For example, I work over an arbitrary field of characteristic 0 rather than just with the complex numbers. Surprisingly, this seems to shorten the proof somewhat. Furthermore, I have added Corollary 2(ii) and the rather uninspiring Lemma 15. In closing, I wish to thank my hosts at the conference, Profs. V. A. Artamonov, Y. A. Bahturin and A. I. Kostrikin, for their invitation and their kind hospitality.

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1. Finite Degrees. The degree of an irreducible module is best understood in terms of polynomial identities. Let K be a field and let R be a K-algebra.

Then R is said to satisfy a polynomial identity if there exists a nonzero polynomial $f(\zeta_1, \zeta_2, \ldots, \zeta_n)$ in the noncommutative polynomial ring $K\langle \zeta_1, \zeta_2, \ldots, \zeta_n, \ldots \rangle$ such that $f(r_1, r_2, \ldots, r_n) = 0$ for all $r_i \in R$. For example, R is commutative if and only if it satisfies the polynomial identity determined by $\zeta_1\zeta_2 - \zeta_2\zeta_1$. More generally, if A is a commutative K-algebra, then the Amitsur-Levitzki theorem (see [3, Theorem 5.1.9]) asserts that the matrix algebra $\mathrm{M}_n(A)$ satisfies no identity of degree $< 2n$, but that it does satisfy the standard identity s_{2n} of degree $2n$. Here

$$s_k(\zeta_1, \zeta_2, \ldots, \zeta_k) = \sum_{\sigma \in \mathrm{Sym}_k} (-1)^\sigma \zeta_{\sigma_1} \zeta_{\sigma_2} \cdots \zeta_{\sigma_k}.$$

Now let R be an arbitrary K-algebra, let V be an irreducible (right) R-module and let $\rho\colon R \to \mathrm{End}_K(V)$ be the corresponding irreducible representation. Then we say that V has finite degree if $\rho(R)$ satisfies a polynomial identity. In this case, since $\rho(R) = \bar{R}$ is a primitive ring, a result of Kaplansky [3, Theorem 5.3.4] implies that $\bar{R} = \mathrm{M}_k(D)$ where $D = \mathrm{End}_R(V)$ is a division algebra finite dimensional over its center $\mathbb{Z}(D) = F$. In particular, $\bar{R}$ is a simple Artinian ring with unique faithful irreducible module V. Furthermore, $\dim_F \bar{R} = n^2$ is the square of a positive integer and we define the degree of V to equal n. Thus the degree of V is neither the dimension of V over D nor its dimension over F. Rather, $\deg V = \dim_E V$ where E is any maximal subfield of D. Note that $D \otimes_F E \cong \mathrm{M}_\ell(E)$, so $\bar{R} \otimes_F E \cong \mathrm{M}_{k\ell}(E) = \mathrm{M}_n(E)$, and the Amitsur-Levitzki theorem yields

Lemma 1. *Let V be an irreducible R-module with corresponding representation ρ. If n is a positive integer, then the following are equivalent.*

(i) $\deg V \le n$.

(ii) $\rho(R)$ satisfies a polynomial identity of degree $2n$.

(iii) $\rho(R)$ satisfies the standard identity s_{2n}.

Because of part (iii) above, this information can be lifted globally to yield

Lemma 2. *Let R be a K-algebra and let $n \ge 1$ be given.*

(i) If R satisfies a polynomial identity of degree $2n$, then all irreducible R-modules have degree $\le n$.

(ii) Conversely, if R is semiprimitive and if all irreducible R-modules have degree $\le n$, then R satisfies the standard identity s_{2n}.

In general, we have no control over the nature of the maximal subfield E in $\rho(R)$. Indeed, the only result of any real interest here is a consequence of Amitsur's trick (see [3, Lemma 7.1.2(iii)]). Namely, suppose that K is algebraically closed and that $\dim_K R < |K|$, where the latter is a strict inequality of cardinal numbers.

Then every irreducible R-module V satisfies $D = \mathrm{End}_K(V) = K$. In particular, if K is the field of complex numbers and if R is a countable dimensional K-algebra, then $\deg V = \dim_K V$ for all such V.

In the following, we use the standard convention that if V is a right R-module, then V is naturally a left module for $\mathrm{End}_R(V)$.

Lemma 3. *Let $R \supseteq S$ be K-algebras, let V be an irreducible right R-module of finite degree, and let W be an irreducible S-submodule of V_S, the restriction of V to S. Then we have*

(i) W has finite degree and $\deg W \leq \deg V$.

(ii) If $\deg W = \deg V$, *then $V = DW$ where* $D = \mathrm{End}_R(V)$.

Proof. Let ρ be the R-representation corresponding to V and let ψ be the S-representation corresponding to W. Then $\rho(R)$ satisfies a polynomial identity, and this identity is also satisfied by its subring $\rho(S)$ and by the latter ring's homomorphic image $\psi(S)$. Thus, by Lemma 1, W has finite degree and indeed $\deg W \leq \deg V$. To study the possibility of equality, we have to take a somewhat more concrete view of V. To this end, write $D = \mathrm{End}_R(V)$, $F = \mathbb{Z}(D)$, and let E be a maximal subfield of D. Then $\rho(R) = \mathrm{M}_k(D)$ where $k = \dim_D V$, and say $D \otimes_F E \cong \mathrm{M}_\ell(E)$. Then $\rho(R) \otimes_F E \cong \mathrm{M}_{k\ell}(E) = \mathrm{M}_n(E)$ where $n = \deg V$. Next, note that DW is a D-subspace of V which is also a right S-module. If $k' = \dim_D DW$ and if $\xi \colon S \to \mathrm{End}_K(DW)$ denotes the corresponding S-representation, then certainly $\xi(S) \subseteq \mathrm{M}_{k'}(D) \otimes_F E \cong \mathrm{M}_{k'\ell}(E) = \mathrm{M}_{n'}(E)$. Thus $\xi(S)$ satisfies the standard identity $s_{2n'}$ and hence so does its homomorphic image $\psi(S)$, obtained by restriction to $W \subseteq DW$. It follows from Lemma 1 that $\deg W \leq n'$. In particular, if $\deg W = \deg V = n$, then $k\ell = n \leq n' = k'\ell$, so $\dim_D V = k \leq k' = \dim_D DW$ and consequently $V = DW$, as required. □

In the remainder of this paper let K be a fixed field (usually of characteristic 0) and let $K[G]$ denote the group algebra of the multiplicative group G over K. Following [6], we say that G has f.r.d (finite representation degree) if all irreducible $K[G]$-modules have finite degree.

Lemma 4. *Some basic properties are as follows.*

(i) Let H be a subgroup of G and let W be an irreducible $K[H]$-module. Then there exists an irreducible $K[G]$-module V such that W is a submodule of V_H, the restriction of V to $K[H]$.

(ii) The f.r.d. condition is inherited by subgroups and homomorphic images.

(iii) A nonabelian free group does not have f.r.d.

Proof. Part (i) is an easy consequence of the fact that $K[H]$ is a left $K[H]$-module direct summand of $K[G]$. See [3, Lemma 6.1.2] for details. Part (ii) for subgroups is an immediate consequence of (i) and Lemma 3(i). Furthermore, if $\bar{G}$ is a homomorphic image of G, then $K[\bar{G}]$ is a homomorphic image of $K[G]$. Thus every irreducible $K[\bar{G}]$-module is naturally an irreducible $K[G]$-module, so $\bar{G}$ clearly inherits the f.r.d. property from G. Finally for (iii), if F is a nonabelian free group, then a result of Formanek (see [3, Corollary 9.2.11]) implies that $K[F]$ is a primitive ring. Since $K[F]$ is not simple, its faithful irreducible module cannot have finite degree. □

In addition, we have

Lemma 5. *Let G have f.r.d. and let* $\operatorname{char} K = 0$. *Then*

(i) Every finitely generated subgroup of G is residually finite.

(ii) $K[G]$ is semiprimitive.

Proof. (i). In view of Lemma 4(ii), we can assume that G is finitely generated. If N denotes the intersection of all normal subgroups of G of finite index, then the goal is to show that $N = 1$. To this end, let g be any nonidentity element of G. Since $\operatorname{char} K = 0$, the group algebra $K[\langle g\rangle]$ of the cyclic subgroup $H = \langle g\rangle$ is semiprimitive and hence there exists an irreducible $K[H]$-module W not annihilated by $1 - g$. It follows from Lemma 4(i) that there exists an irreducible $K[G]$-module V with corresponding representation ρ such that $\rho(g) \neq 1$. But G has f.r.d., so $\rho(G)$ is a finitely generated linear group and hence $\rho(G)$ is residually finite by a result of Mal'cev (see [7, Theorem 4.2]). In particular, since $\rho(g) \neq 1$, it follows that g is not contained in N. Consequently $N = 1$, as required.

(ii) By (i) and [3, Lemma 7.4.4] it follows that every finitely generated group with f.r.d. has a semiprimitive group algebra. [3, Lemma 7.4.1] and Lemma 4(ii) now yield the result in general. □

We close this section with a version of the polynomial identity theorem of Isaacs and Passman (see [3, Corollary 5.3.8]). It is a consequence of the latter result along with Lemmas 2(ii) and 5(ii).

Proposition 1. *Let G be a group and let* $\operatorname{char} K = 0$. *If all irreducible $K[G]$-modules have finite bounded degree, then G has an abelian subgroup of finite index.*

Results of a similar nature for fields of characteristic $p > 0$ will appear in the forthcoming thesis of Behn at the University of Wisconsin-Madison (see also [1]).

2. Some Special Cases. For the remainder of this paper we assume that K is a fixed field and that $\operatorname{char} K = 0$. Recall that a group G has f.r.d. if all irreducible $K[G]$-modules have finite degree. We will freely use the fact that this property is inherited by subgroups and homomorphic images. One of our goals is to show that such groups have a subgroup of finite index which is (finitely generated)-by-abelian. For this, it is first necessary to consider several special cases, specifically certain solvable groups and linear groups. This work is a bit tedious, so we start with groups of class 2. Here, as usual, G' denotes the commutator subgroup of G.

Lemma 6. *Let G have f.r.d.*

(i) If G' is central and cyclic, then G' is finite and $|G\colon \mathbb{Z}(G)| < \infty$.

(ii) If G' is finite, then G has an abelian subgroup of finite index.

Proof. *(i).* Since G' is cyclic and $\operatorname{char} K = 0$, it follows from Lemma 4(i) that $K[G] = R$ has an irreducible module V on which G' acts faithfully. Let ρ denote the corresponding representation and let F denote the center of the primitive ring $\rho(R)$. Since $\deg V < \infty$, we know that F is a field and that $\dim_F \rho(R) < \infty$. Let $Z = \mathbb{Z}(G)$, so that $\rho(Z) \subseteq F$. We claim that if $g_1, g_2, \ldots, g_n \in G$ are in distinct cosets of Z, then $\rho(g_1), \rho(g_2), \ldots, \rho(g_n)$ are F-linearly independent. Indeed, if this is not the case, let $\sum_1^n f_i\rho(g_i) = 0$ be a nontrivial dependence relation with n minimal. By multiplying by $\rho(g_n^{-1})$, we can assume that $g_n = 1$. Certainly $n > 1$, $f_1 \neq 0$ and $g_1 \notin Z$. Thus we can choose $x \in G$ which does not commute with g_1 and, for each i, let us write $g_i^x = z_i g_i$ with $z_i \in G' \subseteq Z$. Multiplying the given linear dependence on the left by $\rho(x^{-1})$ and on the right by $\rho(x)$ yields $\sum_1^n f_i\rho(z_i)\rho(g_i) = 0$, and by subtracting, we obtain

$$0 = \sum_{i=1}^{n} f_i[\rho(z_i) - 1]\rho(g_i) = \sum_{i=1}^{n-1} f_i[\rho(z_i) - 1]\rho(g_i),$$

a shorter dependence relation since $g_n = 1$ implies that $z_n = 1$. Furthermore, $z_1 \neq 1$ and G' acts faithfully on V, so $\rho(z_1) \neq 1$ and $f_1[\rho(z_1) - 1]$ is a nonzero element of F. In other words, this shorter relation is also nontrivial, thereby contradicting the minimality of n and proving the claim. In particular, it follows that $|G\colon Z| \leq \dim_F \rho(R) < \infty$ and hence also that G' is finite.

(ii). If $C = \mathbb{C}_G(G')$, then $|G\colon C| < \infty$ and C' is a finite central subgroup of C. In particular, C' is a finite direct product of cyclic groups and hence C is a finite subdirect product of groups with cyclic central commutator subgroups. By (i), each such factor has a center of finite index, so $|C\colon \mathbb{Z}(C)| < \infty$ and $\mathbb{Z}(C)$ is the required abelian subgroup of G of finite index. □

Next we consider certain metabelian groups.

Lemma 7. *Let G be a metabelian group with f.r.d. Suppose that G' is countable and that the set of normal subgroups of G contained in G' satisfies the maximal condition. Then G has an abelian subgroup of finite index.*

Proof. Let use first just assume that G is a metabelian group with f.r.d. and with a countable commutator subgroup. We proceed in a series of steps.

Step 1. *If G' is periodic and contains no nonidentity finite normal subgroup of G, then G is abelian.*

Proof. Since G' is countable and contains no nontrivial finite normal subgroup of G, [3, Lemma 9.2.6], a slight extension of a result of Formanek and Snider, implies that $K[G]$ has an irreducible module V on which $K[G']$ acts faithfully. In particular, if ρ is the corresponding representation and if F is the center of $\rho(K[G])$, then $K[G'] \cong \rho(K[G']) \subseteq F\rho(K[G'])$ and the latter is a finite dimensional commutative F-algebra. It follows that $K[G']$ has only finitely many idempotents, and since each finite subgroup of G' gives rise to a unique principal idempotent, we conclude that G' is a finite normal subgroup of G. Hence $G' = 1$. □

Step 2. *If G' is torsion free, then G has an abelian subgroup of finite index.*

Proof. Since G' is abelian, it has a free abelian subgroup A with G'/A periodic. Furthermore, G' is countable, so there exists an embedding λ of A into the multiplicative group of the rationals Q and hence into the muliplicative group of K. Of course, λ corresponds to a 1-dimensional irreducible representation of $K[A]$ and, by Lemma 4(i), we can extend this to an irreducible representation of $K[G']$ with corresponding module W. Since G' is abelian, the representation here must be a homomorphism onto some field $L \supseteq K$ and, in particular, λ extends to a homomorphism $\lambda\colon G' \to L^\bullet$. Of course, if $g \in G'$, then $\lambda(g) - g$ annihilates W. Finally, note that G'/A is periodic and $\ker\lambda \cap A = 1$, so $\ker\lambda$ is periodic and hence $\ker\lambda = 1$.

Next, extend W to an irreducible $K[G]$-module V, and let D be the division ring $\operatorname{End}_{K[G]}(V)$. Since $G' \triangleleft G$ and $W \subseteq V$, Clifford's theorem implies that $V_{G'}$, the restriction of V to $K[G']$, is completely reducible with all irreducible submodules G-conjugate to W. Furthermore, $V_{G'}$ is the direct sum of the isotypical components corresponding to these irreducible submodules. In particular, since each isotypical component is a D-subspace of V and since $\dim_D V < \infty$, it follows that only finitely many exist. Furthermore, we know that G permutes these isotypical components, so there is a subgroup H of finite index in G such that H stabilizes the component U corresponding to W.

If $a \in A$ and $x \in H$, then $\lambda(a) - a$ annihilates U and hence so does

$$x^{-1}(\lambda(a) - a)x = \lambda(a) - a^x,$$

since $\lambda(a) \in K$. From the nature of the action of $K[G']$ on W, it follows that $\lambda(a) = \lambda(a^x)$ and hence that $a = a^x$. Thus, H centralizes A. But G' is a torsion free abelian group and G'/A is periodic, so this implies that H also centralizes G'. In particular, since $H' \subseteq G'$, we see that H is a class 2 group with a torsion free commutator subgroup. Furthermore, note that any two elements of H generate a class 2 group with a cyclic commutator subgroup. Thus Lemma 6(i) implies that this commutator subgroup must be finite and hence equal to 1. In other words, H is abelian and G is abelian-by-finite. □

Step 3. *Completion of the proof.*

Proof. Here we assume the full hypothesis of the lemma, namely that the set of normal subgroups of G contained in G' satisfies the maximal condition. Now suppose, by way of contradiction, that G is not abelian-by-finite. Then $G/1$ does not have an abelian subgroup of finite index, so the maximal condition implies that there exists $M \triangleleft G$, $M \subseteq G'$ maximal such that $\bar{G} = G/M$ does not have an abelian subgroup of finite index. Note that $\bar{G}' = G'/M \neq 1$ and that $\bar{G}$ is metabelian. Furthermore, if $\bar{N}$ is any nonidentity normal subgroup of $\bar{G}$ contained in $\bar{G}'$, then $\bar{G}/\bar{N}$ is abelian-by-finite.

If $\bar{G}'$ contains a nontrivial finite normal subgroup $\bar{L}$, then $\bar{G}/\bar{L}$ has an abelian subgroup $\bar{H}/\bar{L}$ of finite index. But then $\bar{H}' \subseteq \bar{L}$ is finite, so $\bar{H}$ has an abelian subgroup of finite index by Lemma 6(ii), a contradiction. Next, suppose that the abelian subgroup $\bar{G}'$ has a nontrivial torsion subgroup $\bar{T}$. Then $\bar{G}/\bar{T}$ has an abelian subgroup $\bar{A}/\bar{T}$ of finite index which we can assume to be normal. Of course, $\bar{A}$ is metabelian with $\bar{A}' \subseteq \bar{T}$ periodic. Furthermore, if $\bar{U}$ is any finite normal subgroup of $\bar{A}$ contained in $\bar{A}'$, then $\bar{U}^{\bar{G}}$ is a finite normal subgroup of $\bar{G}$ contained in $\bar{G}'$, and hence $\bar{U}^{\bar{G}} = 1$. Thus, since $\bar{A}'$ is countable, we conclude from Step 1 that $\bar{A}$ is abelian, again a contradiction. In other words, $\bar{G}'$ is torsion free. But then, by Step 2, $\bar{G}$ is abelian-by-finite, and this final contradiction yields the result. □

We can now quickly prove

Lemma 8. *Let G be a finitely generated solvable group with f.r.d. Then G has an abelian subgroup of finite index.*

Proof. We proceed by induction on the derived length of G. Thus suppose that N is the last nonidentity term in the derived series for G. Then $\bar{G} = G/N$ is finitely generated with smaller derived length. By induction, $\bar{G}$ has an abelian subgroup H/N of finite index. Note that H is also finitely generated and that $H' \subseteq N$ is abelian and countable. Thus H is metabelian, and if $x_1, x_2, \ldots, x_n$ are the generators of H, then H' is a module for the integral group ring $Z[H/H']$, finitely generated by the commutators $[x_i, x_j]$ for $i, j = 1, 2, \ldots, n$. Furthermore, H/H' is a finitely generated abelian group, so $Z[H/H']$ is a commutative Noetherian ring. It follows that H' is a Noetherian $Z[H/H']$-module and since the submodules here are precisely the normal subgroups of H contained in H', we conclude from the preceding lemma that H is abelian-by-finite. Hence so is G. □

As a consequence of the above and a deep result on linear groups, we obtain

Lemma 9. *Let G be a finitely generated group with f.r.d. If V is an irreducible $K[G]$-module with corresponding representation ρ, then $\rho(G)$ is abelian-by-finite.*

Proof. Since $\deg V < \infty$, we know that $\rho(G)$ is a finitely generated linear group. Hence by the Tit's alternative (see [7, Theorem 10.16]), $\rho(G)$ is either solvably-by-finite or it has a nonabelian free subgroup. But the latter possibility cannot occur by Lemma 4(iii), and hence $\rho(G)$ has a solvable subgroup H of finite index. Since H is also finitely generated, the previous lemma implies that H is abelian-by-finite and therefore so is $\rho(G)$. $\square$

At this point, it is convenient to make two simple group theoretic observations.

Lemma 10. *Let G have an abelian subgroup of finite index.*

(i) G has a characteristic abelian subgroup of finite index.

(ii) If G is finitely generated, then G is polycylic-by-finite and hence its subgroups satisfy the maximal condition. Furthermore, every maximal subgroup of G has finite index in the group.

Proof. (i). Let $\mathcal{A}$ denote the nonempty set of abelian subgroups of G of finite index and let B be the characteristic subgroup of G generated by the members of $\mathcal{A}$. If $A_1 \in \mathcal{A}$, then $|B\colon A_1| < \infty$, so it is clear that B is generated by A_1 along with finitely many other elements of $\mathcal{A}$, say $A_2, A_3, \ldots, A_n$. But then, $\bigcap_i A_i$ is a subgroup of finite index in G which is central in B. Thus, $\mathbb{Z}(B)$ is a characteristic abelian subgroup of G of finite index.

(ii). Let A be a normal abelian subgroup of G of finite index. Since G is finitely generated, the same is true of A, and it is clear that G is polycylic-by-finite. Now let M be a maximal subgroup of G. If $M \supseteq A$, then certainly $|G\colon M| < \infty$. On the other hand, if $M \not\supseteq A$, then AM is a subgroup of G properly larger than M, so $AM = G$. Note that $A \cap M \triangleleft M$ since $A \triangleleft G$, and that $A \cap M \triangleleft A$ since A is abelian. Thus $A \cap M \triangleleft AM = G$ and, since $A \cap M \subseteq M$, it suffices to mod out by this normal subgroup and assume that $A \cap M = 1$. In this situation, M is finite, $G = A \rtimes M$, and A has no proper M-stable subgroup. In particular, A has no proper characteristic subgroup and, since A is a finitely generated abelian group, it follows that A is finite and that $|G\colon M| = |A| < \infty$, as required. $\square$

We close this section with Temple's first main result which is proved in a rather beautiful manner. Notice the close relationship between the statement here and that of Lemma 10(ii). As usual, K is a fixed field of characteristic 0, and we let $\omega K[G]$ denote the augmentation ideal of $K[G]$.

Theorem 1 [6]**.** *If G is a nonidentity finitely generated group with f.r.d., then every maximal subgroup of G has finite index in the group. In particular, no proper subgroup of G can be dense in the profinite topology. This means that if L is a subgroup properly smaller than G, then there exists a normal subgroup N of finite index in G with $LN \neq G$.*

Proof. Let H be a maximal subgroup of G and let $\operatorname{core}_G H$ denote the largest normal subgroup of G contained in H. Since $H/\operatorname{core}_G H$ is a maximal subgroup of $G/\operatorname{core}_G H$ with the same index, we can clearly mod out by $\operatorname{core}_G H$ and assume that $\operatorname{core}_G H = 1$. Now G permutes the set Ω of right cosets of H in G and we let $V = K\Omega$ be the corresponding permutation module. If $\alpha \in \Omega$ corresponds to the coset H itself, then certainly $V = \alpha \cdot K[G]$ and $\alpha \cdot \omega K[H] = 0$. Next, let W be the submodule of V given by

$$W = V \cdot \omega K[G] = \alpha \cdot K[G] \cdot \omega K[G] = \alpha \cdot \omega K[G].$$

Then certainly G acts trivially on V/W and $W \neq 0$.

Now suppose that x is any element of $G \setminus H$. Then $G = \langle H, x \rangle$, since H is maximal, and hence

$$\omega K[G] = \omega K[H] \cdot K[G] + (1 - x) \cdot K[G].$$

Thus, since $\alpha \cdot \omega K[H] = 0$, we have

$$W = \alpha \cdot \omega K[G] = \alpha \cdot \omega K[H] \cdot K[G] + \alpha(1 - x) \cdot K[G] = \alpha(1 - x) \cdot K[G].$$

In particular, $W \neq 0$ is a finitely generated $K[G]$-module, so it has a maximal proper submodule U. We claim that G acts faithfully on V/U. Indeed, suppose first that $y \in G \setminus H$ acts trivially on this module. Then $\alpha(1 - y) \in U$ and hence $W = \alpha(1-y) \cdot K[G] \subseteq U$, a contradiction. Thus the kernel of the action is a normal subgroup of G contained in H and consequently it is contained in $\operatorname{core}_G H = 1$.

Let $\rho\colon K[G] \to \operatorname{End}_K(W/U)$ denote the $K[G]$-representation corresponding to the irreducible module W/U. Then Lemma 9 implies that $\rho(G)$ is an abelian-by-finite group. In particular, G has a subgroup A of finite index such that A' acts trivially on W/U. But A' also acts trivially on V/W, so it follows that A'' is trivial on V/U and hence that $A'' = 1$. In other words, A is a finitely generated metabelian group with f.r.d., so A has an abelian subgroup of finite index by Lemma 8. It follows that G also has an abelian subgroup of finite index, and we conclude from Lemma 10(ii) that $|G\colon H| < \infty$.

The observation concerning the profinite topology of G is now immediate. Indeed, suppose that L is a proper subgroup of G. Then the goal is to find a normal subgroup N of finite index in G with $LN \neq G$. Since G is finitely generated, L is contained in a maximal subgroup and thus we can assume that L is maximal. By the above, this implies that $|G\colon L| < \infty$ and, in particular, if we take $N = \operatorname{core}_G L$, then N is a normal subgroup of finite index in G with $LN = L \neq G$, as required. In other words, L does not cover the finite quotient G/N, and therefore L is not dense in the profinite topology. □

3. Cofinite Modules. As usual, let K denote a fixed field of characteristic 0. If G is a finitely generated group with f.r.d., then we know from Lemma 5(i) that G is residually finite. Indeed, G satisfies the very much stronger properties given below. Note that part (ii) uses the fact that G has no proper dense subgroup in the profinite topology.

To avoid unnecessary repetition, we introduce some convenient notation. To this end, let G be an arbitrary group. Then we use $\mathcal{N}(G)$ to denote the set of all normal subgroups of G of finite index. Furthermore, if V is an irreducible $K[G]$-module, then we write $\ker_G V = \{g \in G \mid V(1-g) = 0\}$. Thus $\ker_G V$ is the kernel in G of the representation $\rho\colon K[G] \to \mathrm{End}_K(V)$ associated with V. Hence $\ker_G V \triangleleft G$ and V is the lifting to $K[G]$ of an irreducible module of $K[G/\ker_G V]$.

Lemma 11. *Suppose G is a finitely generated group with f.r.d., let H be a subgroup of G and let $M \in \mathcal{N}(H)$.*

> *(i) There exists $N \in \mathcal{N}(G)$ with $H \cap N \subseteq M$.*
>
> *(ii) If H is finitely generated and if $x \in G \setminus H$, then there exists $N \in \mathcal{N}(G)$ with $H \cap N \subseteq M$ and $x \notin HN$.*

Proof. Let G, H and M be as above. We proceed in a series of three steps.

Step 1. *Part* (i) *holds if G is abelian-by-finite.*

Proof. Let A be a normal abelian subgroup of G of finite index and set $L = HA$ so that $|G\colon L| < \infty$ and L is finitely generated. Note that $M \cap A \in \mathcal{N}(H)$ since both M and $H \cap A$ are contained in this set. Furthermore, $M \cap A \triangleleft A$ since A is abelian. Thus $M \cap A \triangleleft HA = L$ and $H/(M \cap A)$ is a finite subgroup of $L/(M \cap A)$. But $L/(M \cap A)$ is residually finite, by Lemma 5(i), so there exists a normal subgroup of finite index in this group which is disjoint from $H/(M \cap A)$. In other words, there exists $N_1 \in \mathcal{N}(L)$ with $H \cap N_1 \subseteq M \cap A \subseteq M$. Finally, note that $|G\colon L| < \infty$ and $|L\colon N_1| < \infty$, so $|G\colon N_1| < \infty$. Hence $N = \mathrm{core}_G N_1 \in \mathcal{N}(G)$ and, since $H \cap N \subseteq H \cap N_1 \subseteq M$, this part is proved. □

Step 2. *Part* (i) *holds in general.*

Proof. Since H/M is finite, we can let $W_1, W_2, \ldots, W_n$ be the finitely many irreducible $K[H/M]$-modules. By Lemma 4(i), for each i, there exists an irreducible $K[G]$-module V_i which contains W_i as a $K[H]$-submodule. Since G is finitely generated, Lemma 9 implies that $G/\ker_G V_i$ is abelian-by-finite. Hence if $L = \bigcap_1^n \ker_G V_i$, then G/L is also abelian-by-finite. Furthermore, each element of $H \cap L$ acts trivially on the modules W_i, so $H \cap L \subseteq M$ since $K[H/M]$ is semiprimitive. Now $G \supseteq HL \supseteq ML \supseteq L$ and $|HL/ML| \leq |H/M| < \infty$. Thus, since G/L is abelian-by-finite, Step 1 implies that there exists $N \in \mathcal{N}(G)$ with $N \supseteq L$ and $HL \cap N \subseteq ML$. In particular, $H \cap N = H \cap HL \cap N \subseteq H \cap ML = M(H \cap L) = M$, as required. □

Step 3. *Part* (ii) *holds in general.*

Proof. Here we assume that H is finitely generated and that $x \in G \setminus H$. Let $L = \langle H, x\rangle$ so that L is finitely generated and H is a proper subgroup of L. By Step 2, there exist $N_1 \in \mathcal{N}(L)$ with $H \cap N_1 \subseteq M$. Also, by Theorem 1, there exists $N_2 \in \mathcal{N}(L)$ with $HN_2 \neq L$ and hence with $x \notin HN_2$. Set $N_3 = N_1 \cap N_2$ so that $N_3 \in \mathcal{N}(L)$. By Step 2, using $L \subseteq G$ and $N_3 \in \mathcal{N}(L)$, there exists $N \in \mathcal{N}(G)$ with $L \cap N \subseteq N_3$. In particular, $H \cap N = H \cap L \cap N \subseteq H \cap N_3 \subseteq H \cap N_1 \subseteq M$. Furthermore, if $x \in HN$, then $x \in HN \cap L = H(L \cap N) \subseteq HN_3 \subseteq HN_2$, a contradiction. Thus $x \notin HN$ and the lemma is proved. □

The above translates easily to a result on cofinite modules. Specifically, if G is an arbitrary group, we say that V is a cofinite module if V is an irreducible $K[G]$-module with $\ker_G V \in \mathcal{N}(G)$. Thus V is cofinite if and only if it is the lifting to $K[G]$ of an irreducible $K[G/N]$-module for some $N \in \mathcal{N}(G)$.

Now suppose that H is a subgroup of G and that W is an irreducible $K[H]$-module. Then we say that the irreducible $K[G]$-module V extends W if W is a $K[H]$-submodule of V_H, the restriction of V to $K[H]$. Note that if W is given, then Lemma 4(i) asserts that there always exists an irreducible V extending W.

Lemma 12. *Suppose G is a finitely generated group with f.r.d., let H be a subgroup of G and let W be a cofinite $K[H]$-module.*

(i) There exists a cofinite $K[G]$ module V which extends W.

(ii) If H is finitely generated and if $\mathcal{V}$ denotes the set of all cofinite $K[G]$-modules which extend W, then $\bigcap_{V\in\mathcal{V}} \ker_G V \subseteq \ker_H W \subseteq H$.

Proof. Let $M = \ker_H W$ so that, by assumption, $|H : M| < \infty$. By Lemma 11(i), there exists $N \in \mathcal{N}(G)$ with $H\cap N \subseteq M$, and note that HN/N is a subgroup of the finite group G/N. Furthermore, since $H \cap N \subseteq M = \ker_H W$, we see that W can be viewed as an irreducible module for the group algebra of $HN/N \cong H/(H\cap N)$. Thus, by Lemma 4(i), there exists an irreducible $K[G/N]$-module V which extends W. In particular, if we view V as an irreducible $K[G]$-module, then V is a cofinite module which clearly extends W. This proves (i).

Now suppose, in addition, that H is finitely generated, and let x be any element of $G \setminus H$. By Lemma 11(ii), there exists $N_x \in \mathcal{N}(G)$ with $H \cap N_x \subseteq M$ and $x \notin HN_x$. Again, $\bar{H}_x = HN_x/N_x \cong H/(H \cap N_x)$ is a subgroup of the finite group $\bar{G}_x = G/N_x$ and, since $x \notin HN_x$, we see that $\bar{x}$, the image of x in $\bar{G}_x$ is not contained in $\bar{H}_x$. As above, W can be viewed as an irreducible $K[\bar{H}_x]$-module and let $\bar{e}_x$ be the corresponding central idempotent. Since $\bar{e}_x \in K[\bar{H}_x]$ and $\bar{x} \notin \bar{H}_x$, we see that the element $\bar{e}_x(1-\bar{x})$ of $K[\bar{G}_x]$ is not zero. But $K[\bar{G}_x]$ is semiprimitive, so this algebra has an irreducible module V_x with $V_x\bar{e}_x(1 - \bar{x}) \neq 0$. Since $V_x\bar{e}_x \neq 0$, it is clear that $(V_x)_{\bar{H}_x}$, the restriction of V_x to $K[\bar{H}_x]$ must contain an isomorphic copy of W. In other words, V_x extends W. Furthermore, $V_x(1-\bar{x}) \neq 0$ means that $\bar{x} \notin \ker_{\bar{G}_x} V_x$. In particular, if we view V_x as a $K[G]$-module, then V_x is cofinite, V_x extends W, and $x \notin \ker_G V_x$.

To finish part (ii), observe that $\bigcap_{V\in\mathcal{V}} \ker_G V \subseteq \ker_G V_x$ and hence x is not contained in the intersection. But x is an arbitrary element of $G\setminus H$, so it follows that $\bigcap_{V\in\mathcal{V}} \ker_G V \subseteq H$. Finally, if V is any element of $\mathcal{V}$, then since V extends W, we have $\bigcap_{V\in\mathcal{V}} \ker_G V \subseteq H \cap \ker_G V \subseteq \ker_H W$, as required. □

With this, we can prove Temple's main result. While the following appears to apply only to infinitely generated groups, it does in fact yield information on finitely generated groups because such groups could, at least apriori, have infinitely generated subgroups.

Theorem 2 [6]**.** *Let K be a field of characteristic 0 and assume that all irreducible modules for the group algebra $K[G]$ have finite degree. Then G has a finitely generated normal subgroup N such that G/N is abelian-by-finite.*

Proof. For convenience, let $\mathcal{M}$ denote the set of all pairs (H,W) where H is a finitely generated subgroup of G and W is a cofinite $K[H]$-module. We write $(H,W) < (L,V)$ if H is a subgroup of L, V extends W, and $\deg W < \deg V$. Furthermore, we say that (H,W) is degree extendible if at least one pair (L,V) exists with $(H,W) < (L,V)$. The proof proceeds in a series of three steps.

Step 1. *Not every element of $\mathcal{M}$ is degree extendible.*

Proof. Suppose, by way of contradiction, that every $(H,W) \in \mathcal{M}$ is degree extendible. Then we can start with (L_0,V_0), where $L_0 = 1$ and V_0 is the unique irreducible $K[L_0]$-module, and inductively construct an increasing sequence $(L_0,V_0) < (L_1,V_1) < (L_2,V_2) < \cdots$ of elements of $\mathcal{M}$. If we set $L = \bigcup_{i=0}^{\infty} L_i$ and $V = \bigcup_{i=0}^{\infty} V_i$, then it is clear that L is a subgroup of G and that V is an irreducible $K[L[$-module. By assumption, $\deg V = n < \infty$ and thus, if ρ denotes the representation $K[L] \to \mathrm{End}_K(V)$, then $\rho(K[L])$ satisfies a polynomial identity of degree $2n$. Hence the same is true of each $\rho(K[L_i])$. In particular, if $\rho_i\colon K[L_i] \to \mathrm{End}_K(V_i)$ is the representation associated to V_i, then $\rho_i(K[L_i])$ also satisfies a polynomial identity of degree $2n$, since $\rho_i(K[L_i])$ is a homomorphic image of $\rho(K[L_i])$ obtained via restriction. In other words, $\deg V_i \le n$ for all i, and this certainly contradicts the fact that $1 = \deg V_0 < \deg V_1 < \deg V_2 < \cdots$. □

Step 2. *If $(H,W) \in \mathcal{M}$ is not degree extendible, then $N = \ker_H W$ is a finitely generated normal subgroup of G.*

Proof. Since $|H\colon N| < \infty$, it is clear that N is finitely generated. To show that N is normal in G, it suffices to show that $N \triangleleft L$ for all finitely generated groups L containing H. To this end, let L be given and let $\mathcal{V}$ denote the set of all cofinite $K[L]$-modules which extend W. Then $M = \bigcap_{V\in\mathcal{V}} \ker_L V$ is a normal subgroup of L with $M \subseteq N$ by Lemma 12(ii). On the other hand, if $V \in \mathcal{V}$, then $\deg V = \deg W$ by Lemma 3(i), since (H,W) is not degree extendible. Thus, by Lemma 3(ii), $V = DW$ where $D = \mathrm{End}_{K[L]}(V)$, and hence any element of

$K[H]$ which annihilates W also annihilates V. In particular, $\ker_H W \subseteq \ker_L V$ and hence $N = \ker_H W \subseteq \bigcap_{V \in \mathcal{V}} \ker_L V = M$. We conclude that $N = M \triangleleft L$, as required. □

Step 3. *Completion of the proof.*

Proof. By Step 1, we can choose a pair $(H, W) \in \mathcal{M}$ which is not extendible, and by Step 2, we know that $N = \ker_H W$ is a finitely generated normal subgroup of G. Our goal here is to show that $\bar{G} = G/N$ is abelian-by-finite. To this end, note first that W can be viewed as $\bar{W}$, an irreducible $K[\bar{H}]$-module, where $\bar{H}$ is the finite group $\bar{H} = H/N$. If $\bar{e} \in K[\bar{H}]$ is the central idempotent corresponding to $\bar{W}$ and if the degree of $\bar{W}$ is equal to n, then we claim that $K[\bar{G}]$ satisfies the nontrivial generalized polynomial identity $\bar{e} \cdot s_{2n}(\zeta_1, \ldots, \zeta_{2n})$.

For this, let $g_1, \ldots, g_{2n}$ be any $2n$ elements of G and let $L = \langle H, g_1, \ldots, g_{2n} \rangle$, so that L is a finitely generated subgroup of G containing H. Suppose $\bar{V}$ is any cofinite $K[\bar{L}]$-module, where $\bar{L} = L/N$, and let ρ be its corresponding representation. If $\bar{V}$ does not extend $\bar{W}$, then certainly $\rho(\bar{e}) = 0$ and hence $\rho(\alpha) = 0$ where $\alpha = \bar{e} \cdot s_{2n}(\bar{g}_1, \ldots, \bar{g}_{2n})$. On the other hand, if $\bar{V}$ does extend $\bar{W}$, then we know that $\deg \bar{V} \leq \deg \bar{W} = n$, since (H, W) is not degree extendible, and hence $\rho(K[L])$ satisfies s_{2n}. In this case, we have $\rho(s_{2n}(\bar{g}_1, \ldots, \bar{g}_{2n})) = s_{2n}(\rho(\bar{g}_1), \ldots, \rho(\bar{g}_{2n})) = 0$ and hence, again, $\rho(\alpha) = 0$. In other words, α is contained in the kernels of all irreducible representations of $K[\bar{L}]$ corresponding to cofinite modules. But $\bar{L}$ is residually finite, by Lemma 5(i), and each finite homomorphic image of $\bar{L}$ has a semiprimitive group algebra. Thus it follows that the intersection of all such kernels is zero. Consequently, $0 = \alpha = \bar{e} \cdot s_{2n}(\bar{g}_1, \ldots, \bar{g}_{2n})$ and $K[\bar{G}]$ does indeed satisfy this generalized identity. [3, Theorem 5.3.15] now implies that $\bar{G}$ has a finite-by-abelian subgroup of finite index. In particular, since any finite-by-abelian group with f.r.d. is abelian-by-finite, by Lemma 6(ii), we conclude that $\bar{G}$ is also abelian-by-finite and the theorem is proved. □

As a consequence, we see that the general conjecture would follow from an affirmative solution in the case of finitely generated groups. Specifically, we have

Corollary 1 [6]**.** *Let G have f.r.d. and suppose that all finitely generated subgroups of G are abelian-by-finite. Then G is also abelian-by-finite. In particular, a locally solvable group with f.r.d. must have an abelian subgroup of finite index.*

Proof. Let N be as in the previous theorem. Then N is a finitely generated normal subgroup of G and, by hypothesis, N is abelian-by-finite. Hence, by Lemma 10(i), N has a characteristic finitely generated abelian subgroup A of finite index. Now, by Theorem 2, G/N has an abelian subgroup B/N of finite index. In particular, B/A is finite-by-abelian and consequently, by Lemma 6(ii), it is abelian-by finite. In other words, G has a subgroup C of finite index with $C' \subseteq A$. But then C' is a finitely generated abelian group, so this group is countable and satisfies the maximal condition on all its subgroups. With this observation, Lemma 7 implies that C is abelian-by-finite and hence so is G. The final remark concerning locally solvable groups follows immediately from the above and Lemma 8. □

4. Additional Consequences. We close this paper with several more consequences of Theorem 2. As usual, we take K to be a fixed field of characteristic 0, and we freely use the fact that the f.r.d. condition is inherited by subgroups and quotient groups. We first need

Lemma 13. *Let G have f.r.d.*

> *(i) Suppose $N \triangleleft G$ with N and G/N both abelian-by-finite. Then G is abelian-by-finite.*
>
> *(ii) G has a characteristic subgroup A of finite index such that A' is contained in a finitely generated normal subgroup of G.*

Proof. (i). Since N is abelian-by-finite, Lemma 10(i) implies that N has a characteristic abelian subgroup A of finite index. Then, since G/N is abelian-by-finite, we see that G/A has a finite-by-abelian subgroup B/A of finite index. By Lemma 6(ii) and the f.r.d. condition, B/A is abelian-by-finite. In other words, G has a subgroup C of finite index with $C' \subseteq A$. Thus C is a metabelian group with f.r.d., and we conclude from Corollary 1 that C is abelian-by-finite.

(ii). Let $\mathcal{A}$ denote the set of all normal subgroups A of G of finite index such that A' is contained in a finitely generated normal subgroup of G. By Theorem 2, $\mathcal{A}$ is nonempty. Furthermore, since each member of $\mathcal{A}$ has finite index in G, it is clear that $\mathcal{A}$ has a maximal member, say B. We claim that B is the unique maximal member, so that B is characteristic in G. To this end, let $C \in \mathcal{A}$ and say $B' \subseteq N$ and $C' \subseteq M$ where N and M are both finitely generated normal subgroups of G. Then $A = BC$ is a normal subgroup of G of finite index, $Q = NM$ is a finitely generated normal subgroup of G, and $B', C' \subseteq Q$. Let $\bar{A} = AQ/Q$. Then $\bar{B} = BQ/Q$ and $\bar{C} = CQ/Q$ are both normal abelian subgroups of $\bar{A} = \bar{B}\bar{C}$ of finite index, so $\bar{B} \cap \bar{C}$ is a central subgroup of $\bar{A}$ of finite index. In other words, $\bar{A}$ is center-by-finite, so [3, Lemma 4.1.4] implies that $\bar{A}'$ is finite. In particular, if we let $P/Q = \bar{A}'$, then P/Q is finite, so P is a finitely generated normal subgroup of G containing $A' = (BC)'$. We conclude that $BC \in \mathcal{A}$, and the maximality of B implies that $B \supseteq C$. Thus B is indeed the unique maximal member of $\mathcal{A}$, and consequently B is characteristic in G. □

With this, we can prove

Corollary 2 [6]**.** *Let G have f.r.d.*

> *(i) If G is finitely generated, then so is G'.*

(ii) In any case, G has a characteristic subgroup of finite index with a finitely generated commutator subgroup.

Proof. (i). By Lemma 13(ii) applied to G', we know that G' has a characteristic subgroup A of finite index with A' contained in a finitely generated normal subgroup N of G'. Now $A' \triangleleft G$, and G/A' is the extension of the abelian-by-finite group G'/A' by the abelian group G/G'. It therefore follows from Lemma 13(i) that G/A' is abelian-by-finite. Thus, since G is finitely generated, Lemma 10(ii) implies that G/A' is polycyclic-by-finite and hence all its subgroups are finitely generated. In particular, G'/A' is finitely generated and hence so is G'/N since $G' \supseteq N \supseteq A'$. But N is finitely generated and G'/N is finitely generated, so we conclude that G' is indeed finitely generated.

(ii). The argument here is similar. By Lemma 13(ii), G has a characteristic subgroup A of finite index with A' contained in a finitely generated normal subgroup N of G. Furthermore, by Lemma 13(ii) again, A' has a characteristic subgroup B of finite index with B' contained in a finitely generated normal subgroup M of A'. Note that $B' \triangleleft G$ and that G/B' is the extension of the abelian-by-finite group A'/B' by the abelian-by-finite group G/A'. Thus, by Lemma 13(i), G/B' is abelian-by-finite, and hence the same is true of its subgroup N/B'. But N is finitely generated, so Lemma 10(ii) implies that N/B' is polycyclic-by-finite and hence that all its subgroups are finitely generated. In particular, A'/B' is finitely generated and hence so is A'/M since $A' \supseteq M \supseteq B'$. But M is finitely generated and A'/M is finitely generated, so A' is also finitely generated. □

Finally, we mention several observations which may be of use in proving that a finitely generated group with f.r.d. is necessarily abelian-by-finite.

Corollary 3 [6]**.** *Let G be a finitely generated group with f.r.d. and suppose that G is not abelian-by-finite. Then G has a homomorphic image $\bar{G}$ such that*

(i) $\bar{G}$ is not abelian-by-finite.

(ii) All proper homomorphic images of $\bar{G}$ are abelian-by-finite.

(iii) All normal subgroups of $\bar{G}$ are finitely generated.

(iv) No nonidentity normal subgroup of $\bar{G}$ is abelian-by-finite.

Proof. Let $\mathcal{M}$ denote the set of all normal subgroups M of G with G/M not abelian-by-finite. By assumption $1 \in \mathcal{M}$ and hence $\mathcal{M}$ is nonempty. Suppose that $\{M_i \mid i \in \mathcal{I}\}$ is a chain in $\mathcal{M}$ and set $M = \bigcup_{i\in\mathcal{I}} M_i$ so that $M \triangleleft G$. If G/M is abelian-by-finite, then G has a subgroup A of finite index with $A' \subseteq M$. Note that A is finitely generated and hence so is A', by Corollary 2(i). But then $A' \subseteq \bigcup_{i\in\mathcal{I}} M_i$ implies that $A' \subseteq M_i$ for some i and hence G/M_i is abelian-by-finite, a contradiction. Thus $M \in \mathcal{M}$ and Zorn's lemma implies that $\mathcal{M}$ has a maximal member, say N. Setting $\bar{G} = G/N$, it is now clear that (i) and (ii) are satisfied. For (iii), let $\bar{L} = L/N$ be a nontrivial normal subgroup of $\bar{G}$. Then, $\bar{G}/\bar{L}$ is abelian-by-finite, so $\bar{G}$ has a normal subgroup $\bar{B}$ of finite index with $\bar{B}' \subseteq \bar{L}$. Again, both $\bar{B}$ and $\bar{B}'$ are finitely generated, and $\bar{B}' \neq 1$ by (i). Thus, $\bar{G}/\bar{B}'$ is abelian-by-finite, so it is polycyclic-by-finite and hence all its subgroups are finitely generated. In particular, $\bar{L}/\bar{B}'$ is finitely generated and, since $\bar{B}'$ is finitely generated, we conclude that $\bar{L}$ is also. This proves (iii), and part (iv) is immediate from parts (i), (ii) and Lemma 13(i). □

It is remarked in [6] that this result is less powerful than it first appears to be. Indeed, suppose that G is any infinite, finitely generated group. Then, as is well known and easily proved by Zorn's lemma, G has a homomorphic image $\bar{G}$ which is just infinite. This means that $\bar{G}$ is infinite, but all proper homomorphic images of $\bar{G}$ are finite. Now suppose, in addition, that G is periodic. Then $\bar{G}$ is an infinite, finitely generated, periodic group and hence it cannot be abelian-by-finite. In other words, every counterexample to the general Burnside problem gives rise to a group $\bar{G}$ satisfying conditions (i) through (iv) above.

One such Burnside counterexample is the p-group $\mathcal{G}$ constructed by Gupta and Sidki [2]. This group is known to be just infinite and hence $\bar{\mathcal{G}} = \mathcal{G}$. But, $\mathcal{G}$ does not have f.r.d. because it contains an infinite direct product of nonabelian groups. Specifically, we have

Lemma 14. *If G is a group which contains an infinite (weak) direct product of nonabelian groups, then G does not have f.r.d.*

Proof. It is natural to try to prove this result by taking tensor products of irreducible modules. However, this argument can become somewhat unpleasant because K is not assumed to be algebraically closed. Thus we use Theorem 2 or Corollary 2(ii) instead.

In view of Lemma 4(ii), it suffices to assume that $G = G_1 \times G_2 \times \cdots$ is a countably infinite (weak) direct product of the nonabelian groups G_i. Suppose, by way of contradiction, that G has f.r.d. Then, by Corollary 2(ii), G has a subgroup A of finite index with A' finitely generated. In particular, $A' \subseteq G_1 \times G_2 \times \cdots \times G_{r-1}$ for some r and hence, by moding out this direct factor, it follows that the group $H = G_r \times G_{r+1} \times \cdots$ has an abelian subgroup B of finite index. Furthermore, there exists $s > r$ so that $L = G_r \times G_{r+1} \times \cdots \times G_{s-1}$ contains representatives of the finitely many cosets of B. In particular, this means that $H = LB$, and hence $G_s \times G_{s+1} \times \cdots \cong H/L \cong B/(B \cap L)$ is abelian, a contradiction. □

Less interesting, perhaps, is

Lemma 15. *Let G be a group with f.r.d. Then G has an abelian-by-finite subgroup H such that no properly larger subgroup of G is abelian-by-finite.*

Proof. Let $\mathcal{H}$ denote the set of abelian-by-finite subgroups of G. Then $1 \in \mathcal{H}$ so $\mathcal{H}$ is nonempty. Suppose $\{L_i \mid i \in \mathcal{I}\}$ is a chain in $\mathcal{H}$ and set $L = \bigcup_{i\in\mathcal{I}} L_i$. Then L is a subgroup of G and hence, by Corollary 2(ii), L has a subgroup A of finite index with A' finitely generated. Now $A' \subseteq \bigcup_{i\in\mathcal{I}} L_i$ implies that $A' \subseteq L_i$ for some subscript i, so A' is abelian-by-finite. It therefore follows from Lemma 13(i) that A is also abelian-by-finite, and hence the same is true of L. In other words, $L \in \mathcal{H}$, and we can now apply Zorn's lemma to conclude that $\mathcal{H}$ has a maximal member, say H. With this, the lemma is proved. □

Obviously, there is still work to be done on this problem.

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