ON THE COUNTEREXAMPLES TO THE UNIT CONJECTURE FOR GROUP RINGS

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Abstract. We offer two comments on the beautiful papers of Giles Gardam and Alan Murray that yield counterexamples to the Kaplansky unit conjecture. First we discuss the determinants of these units in a certain 4×4 matrix representation of the group ring. Then we explain why there is a doubly infinite family of units in the Murray paper.

Let $\mathfrak{G} = \langle a, b \mid (a^2)^b = a^{-2},\, (b^2)^a = b^{-2} \rangle$. Then we know that $\mathfrak{G}$ is a torsion-free group with a normal abelian subgroup $\mathfrak{H}$ of index 4 and with $\mathfrak{G}/\mathfrak{H}$ a fours group. The paper [G] offers an example of a nontrivial unit in the group algebra $\mathbb{F}_2[\mathfrak{G}]$ where $\mathbb{F}_2 = \mathrm{GF}(2)$. Building on that, [M] offers a doubly infinite family of nontrivial units in $\mathbb{F}_d[\mathfrak{G}]$ for any prime d where $\mathbb{F}_d = \mathrm{GF}(d)$. Of course a unit is nontrivial if it is not a scalar multiple of an element of $\mathfrak{G}$.

Now it is a standard fact that $\mathbb{F}_d[\mathfrak{G}]$ embeds in the 4×4 matrix ring over $\mathbb{F}_d[\mathfrak{H}]$. Indeed write $\mathbb{V} = \mathbb{F}_d[\mathfrak{G}]$. Then $\mathbb{V}$ is a faithful right $\mathbb{F}_d[\mathfrak{G}]$-module via right multiplication and $\mathbb{V}$ is a free left $\mathbb{F}_d[\mathfrak{H}]$-module via left multiplication where the coset representatives $1, a, b, c = ab$ of $\mathfrak{H}$ in $\mathfrak{G}$ yield a free basis for $\mathbb{V}$. Since right and left multiplication commute as operators on $\mathbb{V}$, it follows that $\mathbb{F}_d[\mathfrak{G}]$ embeds in the $\mathbb{F}_d[\mathfrak{H}]$-endomorphisms of $\mathbb{V}$, namely $\mathbf{M}_4(\mathbb{F}_d[\mathfrak{H}])$. Of course, a similar argument holds for any group $\mathfrak{G}$ and any subgroup $\mathfrak{H}$ of finite index, normal or not.

In our situation, $\mathfrak{H}$ is the free abelian group on $x = a^2, y = b^2$ and $z = c^2$. Thus $\mathbb{F}_d[\mathfrak{H}] = \mathcal{L}_d(x, y, z)$, the Laurent polynomial ring in variables x, y, z over $\mathbb{F}_d$, and thus $\mathbb{F}_d[\mathfrak{G}]$ embeds in $\mathbf{M}_4(\mathcal{L}_d(x, y, x))$. Using capital letters for the matrices corresponding to the generators of $\mathfrak{G}$, we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & x^{-1}yz^{-1} \\ 0 & 0 & y^{-1}z & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ y & 0 & 0 & 0 \\ 0 & y^{-1} & 0 & 0 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & x & 0 \\ 0 & x^{-1}z^{-1} & 0 & 0 \\ z & 0 & 0 & 0 \end{bmatrix}$$

$$X = A^2 = \mathrm{diag}(x, x, x^{-1}, x^{-1}) \qquad Y = B^2 = \mathrm{diag}(y, y^{-1}, y, y^{-1})$$

$$Z = C^2 = \mathrm{diag}(z, z^{-1}, z^{-1}, z)$$

Following [M] we choose a prime characteristic d and two integer parameters t and w and, using I for the identity matrix, we define the diagonal matrices

$$H = (I - Z^{1-2t})^{d-2}$$

$$P = (I + X)(I + Y)(Z^t + Z^{1-t})H$$

$$Q = Z^w[(I+X)(X^{-1}+Y^{-1}) + (I+Y^{-1})(I+Z^{2t-1})]H$$
$$R = Z^w[(I+Y^{-1})(X+Y)Z^t + (I+X)(Z^t+Z^{1-t})]H$$
$$S = Z^{2t-1} + (4I+X+X^{-1}+Y+Y^{-1})H$$

These, of course, naturally correspond to elements of $\mathbb{F}_d[\mathfrak{H}]$ and thus

$$U = P + QA + RB + SAB = P + QA + RB + SC$$

corresponds to an element of $\mathbb{F}_d[\mathfrak{G}]$. Indeed, it is shown in [M] that U corresponds to a nontrivial unit of the group ring with inverse corresponding to a specific matrix of the form

$$U' = P' + Q'A + R'B + S'C$$

Note that, if $d = 2$ then $H = I$, but H is a nontrivial polynomial in Z for $d > 2$.

Now using any computer algebra system, it is an easy task to describe U and U' for any set of parameters. However even for relatively small d, these matrices look unbelievably complicated and fill numerous computer screens. But when we multiply UU' and $U'U$ it is satisfying that we obtain the 4×4 identity matrix I. These computations verify the assertions in [G] and [M] at least for the specific set of parameters. What is surprising in these computations, for all the parameters we could check, is that the determinants of U and U' always seem to be equal to 1. It is not clear why this should be, but it is easily provable and we do so below. Note that $\det A = 1$ and $\det B = 1$ so all elements of $\mathfrak{G}$ have matrices of determinant 1.

We remark that for general finite $\mathfrak{G}$ and $\mathfrak{H}$, it is known that the determinant of this matrix representation is related to the group theoretic transfer map.

Proposition 1. *For all parameters d, t, w, we have* $\det U = \det U' = 1$.

Proof. Fix a set of parameters. We ignore the group ring, but rather we work in the 4×4 matrix ring over the Laurent polynomial ring in x, y, z. Now $UU' = I$, so $\det U$ is a unit in $\mathcal{L}_d(x,y,z)$ with inverse $\det U'$. Thus $\det U = fx^iy^jz^k$ for some $0 \neq f \in \mathbb{F}_d$ and integers i, j, k.

Consider the homomorphism $\bar{\ }\colon \mathcal{L}_d(x,y,z) \to \mathcal{L}_d(x,y)$ given by $x \mapsto x$, $y \mapsto y$ and $z \mapsto 1$, and extend this to the corresponding 4×4 matrix rings. Then $\overline{Z} = I$. If $d > 2$, then $\overline{H} = 0$ so $\overline{P} = \overline{Q} = \overline{R} = 0$ and $\overline{S} = I$. Thus $\overline{U} = \overline{A}\overline{B}$ has determinant 1. On the other hand, if $d = 2$ then $\overline{H} = I$ and $\overline{Z} = I$, so $\overline{U}$ is independent of the parameters w and t and it is easy to check (using a computer algebra system, if necessary) that $\det \overline{U} = 1$ in this case also. Since $\det \overline{U}$ is the image of $fx^iy^jz^k$ under $\bar{\ }$, we see that $fx^iy^j = 1$, so $f = 1$ and $i = j = 0$. in other words, $\det U = z^k$.

Now consider the homomorphism $\tilde{\ }\colon \mathcal{L}_d(x,y,z) \to \mathcal{L}_d(z)$ given by $x \mapsto -1$, $y \mapsto -1$ and $z \mapsto z$, and extend this to the corresponding 4×4 matrix rings. Then $\widetilde{X} = \widetilde{X}^{-1} = \widetilde{Y} = \widetilde{Y}^{-1} = -I$, so P, Q and R map to 0 and $\widetilde{S} = \widetilde{Z}^{2t-1}$. It follows that $\widetilde{U} = \widetilde{Z}^{2t-1}\widetilde{C}$ has determinant 1. But $\det \widetilde{U}$ is the image of z^k under $\tilde{\ }$, so we see that $z^k = 1$ and $k = 0$, as required. □

Now let us return to the group algebra and let u_0 be the Murray nontrivial unit given by $t = 0$ and $w = 0$. Then $u_0 = p_0 + q_0a + r_0b + s_0ab$ where $h_0 = (1-z)^{d-2}$

$$p_0 = (1+x)(1+y)(1+z)h_0$$
$$q_0 = [(1+x)(x^{-1}+y^{-1}) + (1+y^{-1})(1+z^{-1})]h_0$$
$$r_0 = [(1+y^{-1})(x+y) + (1+x)(1+z)]h_0$$
$$s_0 = z^{-1} + (4+x+x^{-1}+y+y^{-1})h_0$$

The next question to ask is why is there a doubly infinite family of such nontrivial units. The answer here is fairly easy, namely these units correspond to a doubly infinite family of endomorphisms of $\mathfrak{G}$. Specifically

Proposition 2. *Fix the characteristic d and let u be the nontrivial unit of $\mathbb{F}_d[\mathfrak{G}]$ corresponding to the parameters t and w. Then $u = z^t\sigma(u_0)$ where σ is the endomorphism of $\mathbb{F}_d[\mathfrak{G}]$ determined by the group endomorphism $\sigma\colon \mathfrak{G} \to \mathfrak{G}$ given by $a \mapsto z^w z^{-t} a$ and $b \mapsto z^w b$.*

Proof. Let t and w be integers and define group elements in $\mathfrak{G}$ by $\bar{a} = z^w z^{-t} a$ and $\bar{b} = z^w b$. Since a and b invert z by conjugation, it follows that $\bar{a}^2 = a^2 = x$ and $\bar{b}^2 = b^2 = y$. Furthermore, $\bar{c} = \bar{a}\bar{b} = (z^w z^{-t} a)(z^w b) = z^{-t} c$ so $\bar{c}^2 = z^{1-2t} = \bar{z}$. Now $(\bar{a}^2)^{\bar{b}} = (\bar{a})^{-2}$ and $(\bar{b}^2)^{\bar{a}} = (\bar{b})^{-2}$, so it follows from the definition of $\mathfrak{G}$ that there exists a homomorphism $\sigma\colon \mathfrak{G} \to \mathfrak{G}$ with $\sigma(a) = \bar{a}$ and $\sigma(b) = \bar{b}$. Note that $\sigma(x) = \sigma(a)^2 = x$, $\sigma(y) = \sigma(b)^2 = y$, $\sigma(c) = \bar{c} = z^{-t} c$ and $\sigma(z) = \bar{c}^2 = z^{1-2t}$. It follows that $\sigma\colon \mathfrak{H} \to \mathfrak{H}$ is one-to-one, but not necessarily onto, and then $\sigma\colon \mathfrak{G} \to \mathfrak{G}$ is also one-to-one, but not necessarily onto. Of course σ extends to an algebra homomorphism $\sigma\colon \mathbb{F}_d[\mathfrak{G}] \to \mathbb{F}_d[\mathfrak{G}]$. In particular, σ sends units to units and, since σ is one-to-one on $\mathfrak{G}$, it sends nontrivial units to nontrivial units.

Now let $u = p + qa + rb + sc$ be the unit associated with t and w. We compute $z^t\sigma(u_0)$ as follows. First $h_0 = (1-z)^{d-2}$ so $\sigma(h_0) = (1 - z^{1-2t})^{d-2} = h$. Next $p_0 = (1+x)(1+y)(1+z)h_0$ so

$$z^t\sigma(p_0) = z^t(1+x)(1+y)(1+z^{1-2t})h = p$$

and $q_0 = [(1+x)(x^{-1}+y^{-1}) + (1+y^{-1})(1+z^{-1})]h_0$ so

$$z^t\sigma(q_0 a) = z^t z^w z^{-t}[(1+x)(x^{-1}+y^{-1}) + (1+y^{-1})(1+z^{2t-1})]ha = qa$$

Similarly $r_0 = [(1+y^{-1})(x+y) + (1+x)(1+z)]h_0$ so

$$z^t\sigma(r_0 b) = z^t z^w[(1+y^{-1})(x+y) + (1+x)(1+z^{1-2t})]hb = rb$$

and $s_0 = z^{-1} + (4 + x + x^{-1} + y + y^{-1})h_0$ so

$$z^t\sigma(s_0 c) = z^t z^{-t}[z^{2t-1} + (4 + x + x^{-1} + y + y^{-1})h]c = sc$$

We conclude that $z^t\sigma(u_0) = u$ and the proposition is proved. □

Of course, $\mathfrak{G}$ admits automorphisms that permute the cosets $\mathfrak{H}a$, $\mathfrak{H}b$ and $\mathfrak{H}c$ transitively. Furthermore, there are additional endomorphisms that fix the cosets of $\mathfrak{H}$, and all of these yield additional nontrivial units in $\mathbb{F}_d[\mathfrak{G}]$. Finally a close look at the last paragraph of the proof of [M, Theorem 3] shows that if we define

$$\begin{aligned} h_0(z) &= \frac{1-z^{nd}}{(1-z)^2} = \frac{(1-z^d)}{(1-z)^2}(1 + z^d + z^{2d} + \cdots + z^{(n-1)d}) \\ &= (1-z)^{d-2}(1 + z^d + z^{2d} + \cdots + z^{(n-1)d}) \end{aligned}$$

for any integer $n \geq 1$, then the expression for h in Theorem 3 can be replaced by

$$h = h_0(z^{1-2t}) \in \mathbb{F}_d[\mathfrak{G}]$$

In this way, for any fixed characteristic $d > 0$, by varying n, t, and w, we obtain a triply infinite family of counterexamples. Note that Propositions 1 and 2 above apply equally well to this more general situation although a small amount of additional work is needed when $d = 2$ in Proposition 1. Namely here we must observe that $\overline{H}$ could equal either 0 or I depending on whether n is even or odd.

References

[G] Gardam, Giles, *A Counterexample to the Unit Conjecture for Group Rings*, arXiv: 2102.11818v3 [math.GR] 14 Apr 2021.

[M] Murray, Alan G., *More Counterexamples to the Unit Conjecture for Group Rings*, arXiv: 2106.02147v1 [math.RA] 3 Jun 2021.

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