AFFINE NOETHERIAN ALGEBRAS AND EXTENSIONS OF THE BASE FIELD II

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This paper is dedicated to our friend Richard Resco who died way too young.

Abstract. The paper [RS], written many years ago by Richard Resco and Lance Small, constructed an example in prime characteristic of a finitely generated Noetherian K-algebra that does not necessarily remain Noetherian under field extensions. After that work was completed, Yuri Medvedev, then at the University of Ottawa, visited Resco at the University of Oklahoma, read the paper, and suggested an approach using Jordan algebras that would yield a similar example in characteristic 0. This was announced in [RS] but to our knowledge the details were never published nor even entirely verified. We take the opportunity to do this now, but stress that the key sequence $\{a_n \mid n = 1, 2, 3, \ldots\}$ and its relevant properties are due to Medvedev. Also Resco suggested avoiding Jordan algebras and working directly and ring theoretically with the Grassman algebra. This is the approach we take.

Let F be a field of any characteristic and let

$$E = F[t_1, t_2, t_3, \ldots \mid t_i t_j = -t_j t_i \text{ for all } i \neq j].$$

Thus E is the Grassman algebra over F in the variables t_i. Since E is the ascending union of the Ore extensions $F[t_1, t_2, \ldots, t_{n-1}][t_n; \sigma]$, the ring is clearly a domain and also locally Noetherian. Note that $Z = F[t_1^2, t_2^2, t_3^2, \ldots]$ is a central subring of E so we can certainly localize E at the nonzero elements of Z. In this way, we obtain the ring $D = KE$ where K is the central subfield given by $K = F(t_1^2, t_2^2, t_3^2, \ldots)$, the ring of quotients of Z. We see easily that D is a K-algebra with basis $\mathfrak{B}$ consisting of the identity 1 and all products of the form $\beta = t_{i_1} t_{i_2} \cdots t_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. Note that this β has t-degree k and let us write i_k for its height. Now for any integer n there are just finitely many basis elements of height $\leq n$, so it follows that the subalgebra $K \cdot F[t_1, t_2, \ldots, t_n]$ is finite dimensional over K. Thus this subalgebra is a division ring and consequently so is D.

Now define the sequence of elements $a_1, a_2, a_3, \ldots$ in D by $a_1 = t_2$ and $a_n = t_{n+1} - (t_n^2/t_{n-1}^2) \cdot t_{n-1}$ for all $n > 1$. Observe that $t_n^2/t_{n-1}^2 \in K$.

Lemma 1. *With the above notation we have*

i. $t_n a_n + a_n t_n = 0$ *for all* $n \geq 1$.
ii. $t_n a_m + a_m t_n + t_m a_n + a_n t_m = 0$ *for all* $n \neq m$.

Proof. (i) This is clear since $t_n t_{n-1} + t_{n-1} t_n = 0$ and $t_n t_{n+1} + t_{n+1} t_n = 0$.

(ii) If $|n - m| \geq 2$ then $t_n a_m + a_m t_n = 0$ and $t_m a_n + a_n t_m = 0$ so the identity holds. Finally, assume that $|n - m| = 1$ and, by symmetry, say $n < m$. Then

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$m = n+1$ and hence $t_m a_n + a_n t_m = 2t_m^2$. Note that this holds even when $n = 1$. Furthermore, since $t_n a_m + a_m t_n = -2t_m^2$, the result follows. □

We now use the a_n-sequence to define a K-linear map $\delta\colon D \to D$. For this, we need only specify the values of δ on the basis elements in $\mathfrak{B}$ and to start with we set $\delta(1) = 0$ and hence $\delta(K) = 0$. Next if β is the basis element $\beta = t_{i_1} t_{i_2} \cdots t_{i_k}$ with $i_1 < i_2 < \cdots < i_k$, then $\delta(\beta)$ is the sum of k products, where the product corresponding to i_j replaces t_{i_j} in β with a_{i_j}. Thus, for example, $\delta(t_{i_1}) = a_{i_1}$, $\delta(t_{i_1} t_{i_2}) = a_{i_1} t_{i_2} + t_{i_1} a_{i_2}$ and

$$\delta(t_{i_1} t_{i_2} t_{i_3}) = a_{i_1} t_{i_2} t_{i_3} + t_{i_1} a_{i_2} t_{i_3} + t_{i_1} t_{i_2} a_{i_3}.$$

Obviously, δ looks like a derivation with $t_n \mapsto a_n$ and indeed we have

Lemma 2. *The map δ, as defined above, is a K-derivation on D.*

Proof. We need to prove that

$$\delta(\alpha\beta) = \delta(\alpha)\beta + \alpha\delta(\beta) \qquad (*)$$

for all $\alpha, \beta \in D$. Since δ is a K-linear function, it suffices to assume that α and β are basis elements in $\mathfrak{B}$. Indeed, we can assume that $\alpha \in \widetilde{\mathfrak{B}}$, the set of K-multiples of $\mathfrak{B}$. Notice that the t-degree and height functions extend naturally to $\widetilde{\mathfrak{B}}$. Clearly the result holds if $\alpha \in K$ or if $\beta = 1$, so assume that neither of these occurs. Then we can write $\alpha = \alpha' t_m$, where t_m is the last factor in the product α, and to start with, let us assume that $\beta = t_n$. In this situation, there are three cases to consider according to the relative sizes of m and n. We proceed by induction on the t-degree of α, but only use induction to deal with the third case.

First if $n > m$, then $\alpha\beta = \alpha' t_m t_n$ is a K-multiple of a basis element, written in its standard product form. The definition of δ immediately implies that $(*)$ holds.

Next suppose that $m = n$. Then since $\alpha \in \widetilde{\mathfrak{B}}$, we have $\delta(\alpha) = \delta(\alpha' t_m) = \delta(\alpha')t_m + \alpha' a_m$. Thus

$$\delta(\alpha)\beta + \alpha\delta(\beta) = [\delta(\alpha')t_m + \alpha' a_m]t_n + (\alpha' t_m)a_n = \delta(\alpha')t_m t_n$$

since $a_m t_n + t_m a_n = 0$ by Lemma 1(i). But $t_m t_n = t_n^2 \in K$ so $\delta(\alpha')t_m t_n = \delta(\alpha' t_m t_n) = \delta(\alpha\beta)$, as required.

For the third case, we assume $m > n$ and recall that $\alpha = \alpha' t_m \in \widetilde{\mathfrak{B}}$ implies that $\delta(\alpha) = \delta(\alpha')t_m + \alpha' a_m$. Furthermore, we have $\alpha\beta = \alpha' t_m t_n = (-\alpha' t_n)t_m$ and notice that $-\alpha' t_n \in \widetilde{\mathfrak{B}}$ has height $< m$. Thus by definition of δ we get

$$\delta(\alpha\beta) = \delta((-\alpha' t_n)t_m) = \delta(-\alpha' t_n)t_m + (-\alpha' t_n)a_m.$$

Since $t\text{-}\deg \alpha' < t\text{-}\deg \alpha$, induction implies that $\delta(-\alpha' t_n) = \delta(-\alpha')t_n - \alpha' a_n$ and hence the above yields

$$\begin{aligned}
\delta(\alpha\beta) &= [\delta(-\alpha')t_n - \alpha' a_n]t_m - (\alpha' t_n)a_m \\
&= \delta(-\alpha')t_n t_m - \alpha'[a_n t_m + t_n a_m] \\
&= \delta(\alpha')t_m t_n + \alpha'[a_m t_n + t_m a_n]
\end{aligned}$$

by Lemma 1(ii). In particular

$$\delta(\alpha\beta) = [\delta(\alpha')t_m + \alpha' a_m]t_n + (\alpha' t_m)a_n = \delta(\alpha)\beta + \alpha\delta(\beta),$$

and the $\beta = t_n$ case is proved.

Finally we consider the remaining possibilities for $\beta \in \mathfrak{B}$ and here we proceed by induction on $t\text{-}\deg \beta$. As we have observed, $(*)$ holds for $t\text{-}\deg \beta = 0$ and also, by the above, for t-deg $\beta = 1$. Thus suppose t-deg $\beta \geq 2$ and write $\beta = t_n\beta'$. Then t-deg t_n and t-deg β' are both $<$ t-deg β and thus

$$\begin{aligned}\delta(\alpha\beta) = \delta(\alpha t_n\beta') &= \delta(\alpha t_n)\beta' + \alpha t_n\delta(\beta') = [\delta(\alpha)t_n + \alpha a_n]\beta' + \alpha t_n\delta(\beta')\\ &= \delta(\alpha)\beta + \alpha[a_n\beta' + t_n\delta(\beta')] = \delta(\alpha)\beta + \alpha\delta(\beta).\end{aligned}$$

The induction step is proved. □

Since δ is a derivation, its kernel is the set of δ constants, and we have

Lemma 3. $\ker\delta = K$.

Proof. We already know that $\delta(K) = 0$. Suppose by way of contradiction that $\ker\delta > K$ and choose a nonzero element $\alpha \in \ker\delta \setminus K$. Write this element in terms of the basis $\mathfrak{B}$ and suppose n is the largest height of the basis elements that occur. Then $\alpha = \sum_i k_i\beta_i + \sum_j z_j\gamma_j$ where k_i and z_j are nonzero elements of K, the basis elements β_i have height $< n$ and the basis elements γ_j have height n. Furthermore write each $\gamma_j = \gamma'_j t_n$, where the γ'_j are distinct basis elements of lower height. Then viewing the equation $\delta(\alpha) = 0$ modulo the subspace $K[t_1, t_2, \ldots, t_n]$ we clearly get $\big(\sum_j z_j\gamma'_j\big)t_{n+1} \equiv 0$. Thus $\sum_j z_j\gamma'_j = 0$ and this contradicts the fact that the γ'_j are distinct basis elements and the coefficients z_j are nonzero elements of K. □

We now use the K-algebra D and the K-derivation δ to construct the ring $R = D[x;\delta]$ of formal differential operators over D. See [GW, Chapter 2] for basic properties of such skew polynomial rings. To start with, R is the set of all formal D-polynomials $\sum_i d_ix^i$ with the powers of x written on the right, by convention. Thus we can speak about the x-degree of a polynomial and also about its constant term. Of course, the ring theoretic structure of R is determined by $xd - dx = \delta(d)$ for all $d \in D$ and $x^i{\cdot}x^j = x^{i+j}$. It is shown in [GW, page 27] that R is indeed a ring and in our situation, K is central in R, so R is a K-algebra. Furthermore, by [GW, page 37], we have

$$x\text{-deg}\, rs = x\text{-deg}\, r + x\text{-deg}\, s \tag{$**$}$$

for all $r, s \in R$. Thus R is a domain and the only units of R are in D. In particular, every subfield of R is contained in D.

Since R is a ring, let $[r, s] = rs - sr$ denote the corresponding Lie product. Then we know that the maps $[r, -]$ and $[-, s]$ are both derivations on R. If $c(r)$ denotes the constant term of the polynomial r, then of course c is a K-linear function and $c(Rx) = 0$. Furthermore, for all $d \in D$ and $n \geq 1$, we have

$$c(x^nd) = c([x^n, d]) = \sum_{i=1}^{n} c\big(x^{i-1}[x, d]x^{n-i}\big) = c(x^{n-1}\delta(d)).$$

By induction, $c(x^nd) = c([x^n, d]) = \delta^n(d)$.

Theorem 4. *Let $R = D[x;\delta]$ be as above. Then*

i. *R is a domain and a principal left and right ideal ring. In particular, R is both left and right Noetherian.*
ii. *R is generated as a K-algebra by $1, t_1$ and x.*
iii. *K is the center of R.*
iv. *R is a simple ring.*

Proof. (i) This is standard since D is a division ring; see [GW, Theorem 2.8].

(ii) Let $K\langle 1, t_1, x\rangle$ denote the K-subalgebra of R generated by $1, t_1$ and x. If $K\langle 1, t_1, x\rangle$ contains $K[t_1, t_2, \ldots, t_n, x]$ then this subalgebra also contains $xt_n - t_n x = a_n$. Thus it contains t_{n+1} and hence the entire subalgebra $K[t_1, t_2, \ldots, t_{n+1}, x]$. By induction, $K\langle 1, t_1, x\rangle \supseteq R$, as required.

(iii) We know that K is central in R, so we just need the opposite inclusion. To this end, let $r = \sum_{i=0}^{n} d_i x^i$ be central in R. Then

$$0 = [x, r] = \sum_i [x, d_i] x^i = \sum_i \delta(d_i) x^i.$$

Thus $\delta(d_i) = 0$ for all i and hence $d_i \in K$ by the preceding lemma. Next since $[t_1, r] = 0$ we have

$$0 = c([t_1, r]) = \sum_{i=1}^{n} d_i c([t_1, x^i]) = \sum_{i=1}^{n} d_i \delta^i(t_1).$$

The nature of the a-sequence clearly implies that the various $\delta^i(t_1)$ are linearly independent over K and hence $d_1, d_2, \ldots, d_n$ are all 0. Thus $r = d_0 \in K$.

(iv) Let $\mathcal{I}$ be a nonzero 2-sided ideal of R and choose $0 \neq r \in \mathcal{I}$ to have minimal x-degree. Since D is a division ring and $\mathcal{I}$ is closed under left multiplication by D, we can assume that r is monic, so that its leading coefficient is 1. Now $\mathcal{I} \triangleleft R$ implies that $[x, r] \in \mathcal{I}$ and then $(**)$ easily implies that $x\text{-deg}\,[x, r] < x\text{-deg}\, r$. Thus $[x, r] = 0$. Similarly $[t_1, r] \in \mathcal{I}$ and $x\text{-deg}\,[t_1, r] < x\text{-deg}\, r$. Thus $[t_1, r] = 0$ and r commutes with both x and t_1. By (ii) above, r is central in R and hence by (iii), $r \in K$. Thus $r = 1$, $\mathcal{I} = R$ and R is simple. □

The following key lemma allows us to prove that certain extensions of R are not Noetherian. Notice the amusing right-left shift.

Lemma 5. *Let $T \supseteq S$ be rings.*

i. Suppose S has a complement in T as a right S-module. If S is not left Noetherian, then neither is T.

ii. Suppose S has a complement in T as a left S-module. If S is not right Noetherian, then neither is T.

Proof. We prove (i). Write $T = S \oplus J$ where J is the given right S-module complement for S. If $\mathcal{L}$ is a left ideal of S, then $T\mathcal{L}$ is a left ideal of T extending $\mathcal{L}$. Furthermore, $T\mathcal{L} = (S \oplus J)\mathcal{L} = \mathcal{L} \oplus J\mathcal{L}$ with $J\mathcal{L} \subseteq J$. Thus $T\mathcal{L} \cap S = \mathcal{L}$ and it follows that any proper chain of left ideals of S gives rise to a proper chain of left ideals of T. In particular, if S is not left Noetherian, then neither is T. The proof of (ii) is of course similar. □

Now let L be the subring of the division ring D generated by the central subfield K and the infinitely many products $t_1t_2, t_3t_4, t_5t_6, t_7t_8, \ldots$. Since the latter generators commute and have squares in K, it follows that L is a subfield of D and that L/K is algebraic of infinite dimension. Recall that R is a finitely generated K-algebra that is both right and left Noetherian. In the following we show that the extended algebra $R \otimes_K L$ is neither right nor left Noetherian.

Theorem 6. *Let R and L be as above. Then $R \otimes_K L$ and $R \otimes_K R$ are neither right nor left Noetherian.*

Proof. By [V, Theorem 11], the structure of L implies that $L \otimes_K L$ is not Noetherian. Furthermore, since $R \supseteq L$ and L is a field, it is clear that L has a complement in R as a left L-module and also as a right L-module. Thus $R \otimes_K L \supseteq L \otimes_K L$ and $L \otimes_K L$ has a complement in $R \otimes_K L$ as a left $L \otimes_K L$-module and also as a right $L \otimes_K L$-module. Since $L \otimes_K L$ is not Noetherian, the previous lemma implies that $R \otimes_K L$ is neither right nor left Noetherian. A similar argument applies to $R \otimes_K R$ using the inclusion $R \otimes_K R \supseteq L \otimes_K L$. □

We note that the above examples answer in the negative the questions posed in [GW, Appendix #15 and #16] for fields of any characteristic.

Finally in the proof of the preceding theorem we worked with a specific subfield of R. But there are certainly numerous other possibilities available. Indeed it is easy to see that any maximal subfield of R satisfies the necessary conditions of [V, Theorem 11], namely that it is an algebraic extension of infinite dimension over K.

On the other hand, for our specific field L above, we can offer an elementary proof that $L \otimes_K L$ is not Noetherian without using the more difficult paper [V]. Indeed, let us write the generators of L as $u_i = t_{2i-1}t_{2i}$ for $i = 1, 2, 3, \ldots$ and notice that any product of distinct u_i's is a member of the K-basis $\mathfrak{B}$ of D. From this it follows that for any choice of $\pm$ signs and any $m \geq 1$, the products $\prod_{i=1}^{m}(u_i \otimes 1 \pm 1 \otimes u_i)$ in $L \otimes_K L$ are nonzero. Now consider the ideal I of $L \otimes_K L$ generated by the various $w_i = u_i \otimes 1 - 1 \otimes u_i$. Since $u_i^2 \in K$, it follows that the ith generator w_i is annihilated by $\overline{w}_i = u_i \otimes 1 + 1 \otimes u_i$. In particular, if I were finitely generated, then it would be generated by $w_1, w_2, \ldots, w_n$ for some n, and hence I would be annihilated by the product $\alpha = \prod_{i=1}^{n} \overline{w}_i$. But this is not the case because, as we have seen above, α does not annihilate w_{n+1}. We conclude that I is not finitely generated and consequently $L \otimes_K L$ is not Noetherian.

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