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BURNSIDE'S THEOREM FOR HOPF ALGEBRAS

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ABSTRACT. A classical theorem of Burnside asserts that if χ is a faithful complex character for the finite group G, then every irreducible character of G is a constituent of some power χ^n of χ. Fifty years after this appeared, Steinberg generalized it to a result on semigroup algebras $K[G]$ with K an arbitrary field and with G a semigroup, finite or infinite. Five years later, Rieffel showed that the theorem really concerns bialgebras and Hopf algebras. In this note, we simplify and amplify the latter work.

Let K be a field and let A be a K-algebra. A map $\Delta: A \to A \otimes A$ is said to be a comultiplication on A if Δ is a coassociative K-algebra homomorphism. For convenience, we call such a pair (A, Δ) a b-algebra. Admittedly, this is rather nonstandard notation. One is usually concerned with bialgebras, that is, algebras which are endowed with both a comultiplication Δ and a counit $\epsilon: A \to K$. However, semigroup algebras are not bialgebras in general, and the counit rarely comes into play here. Thus it is useful to have a name for this simpler object.

Now a b-algebra homomorphism $\theta: A \to B$ is an algebra homomorphism which is compatible with the corresponding comultiplications, and the kernel of such a homomorphism is called a b-ideal. It is easy to see that I is a b-ideal of A if and only if $I \triangleleft A$ with $\Delta(I) \subseteq I \otimes A + A \otimes I$. Of course, the b-algebra structure can be used to define the tensor product of A-modules. Specifically, if V and W are left A-modules, then A acts on $V \otimes W$ via $a(v \otimes w) = \Delta(a)(v \otimes w)$ for all $a \in A$, $v \in V$, $w \in W$. Notice that if I is a b-ideal of A, then the set of all A-modules V with $\operatorname{ann}_A V \supseteq I$ is closed under tensor product. Conversely, we have

Proposition 1. *Let A be a b-algebra and let $\mathcal{F}$ be a family of A-modules closed under tensor product. Then*

$$I = \bigcap_{V \in \mathcal{F}} \operatorname{ann}_A V$$

is a b-ideal of A.

Proof. Certainly I is an ideal of A. Now let $X = \oplus\sum_{V \in \mathcal{F}} V$ be the direct sum of the modules in $\mathcal{F}$. Then X is an A-module and $\operatorname{ann}_A X = \bigcap_{V \in \mathcal{F}} \operatorname{ann}_A V =$

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I. Furthermore, since $X \otimes X = \oplus\sum_{V, W \in \mathscr{F}} V \otimes W$ and since each $V \otimes W \in \mathscr{F}$, it follows that I annihilates $X \otimes X$. In other words,

$$\Delta(I) \subseteq \operatorname{ann}_{A \otimes A} X \otimes X = I \otimes A + A \otimes I$$

and I is a b-ideal of A. □

The assumption that $\mathscr{F}$ is closed under tensor product can be weakened somewhat in the above. Indeed, suppose that for each V, $W \in \mathscr{F}$ there exists $U \in \mathscr{F}$ with $\operatorname{ann}_A U \subseteq \operatorname{ann}_A V \otimes W$. Then certainly $I \subseteq \operatorname{ann}_A U$ annihilates $V \otimes W$, so I annihilates $X \otimes X$ and hence I is a b-ideal of A.

Now if (A, Δ, ϵ) is a bialgebra with counit ϵ, then I is a bi-ideal of A if and only if it is a b-ideal with $\epsilon(I) = 0$. Furthermore, we can trivially guarantee that the ideal I of the previous proposition satisfies $\epsilon(I) = 0$ by including the principal module K_ϵ in the set $\mathscr{F}$. Thus we have

Proposition 1*. *Let A be a bialgebra and let $\mathscr{F}$ be a family of A-modules closed under tensor product. If $K_\epsilon \in \mathscr{F}$, then*

$$I = \bigcap_{V \in \mathscr{F}} \operatorname{ann}_A V$$

is a bi-ideal of A.

Since the coassociativity of Δ guarantees that the tensor product of A-modules is associative, it makes sense to define the nth tensor power of V by

$$V^{\otimes n} = V \otimes V \otimes \cdots \otimes V \quad (n \text{ times})$$

for all $n \geq 1$. Here, $V^{\otimes 1} = V$ and $V^{\otimes m} \otimes V^{\otimes n} = V^{\otimes(m+n)}$ for all $m, n \geq 1$. It is now a simple matter to prove the following result of [Ri].

Corollary 2. *Let A be a b-algebra and let V be an A-module. If $\operatorname{ann}_A V$ contains no nonzero b-ideal, then $\mathscr{T}(V) = \oplus\sum_{n=1}^{\infty} V^{\otimes n}$ is a faithful A-module.*

Proof. $\mathscr{F} = \{V^{\otimes n} \mid n = 1, 2, \ldots\}$ is a set of A-modules which is clearly closed under tensor product. Thus, by Proposition 1,

$$I = \bigcap_{n=1}^{\infty} \operatorname{ann}_A V^{\otimes n} = \operatorname{ann}_A \mathscr{T}(V)$$

is a b-ideal of A. But $I \subseteq \operatorname{ann}_A V^{\otimes 1} = \operatorname{ann}_A V$, so the hypothesis implies that $I = 0$ and hence that $\mathscr{T}(V)$ is faithful. □

If A is a bialgebra, then one usually defines $V^{\otimes 0}$ to equal K_ϵ, since the latter module behaves like the identity element under tensor product. Thus we have

Corollary 2*. *Let A be a bialgebra and let V be an A-module. If $\operatorname{ann}_A V$ contains no nonzero bi-ideal, then $\mathscr{T}^*(V) = \oplus\sum_{n=0}^{\infty} V^{\otimes n}$ is a faithful A-module.*

Let V be an A-module. If J is an ideal of A contained in $\operatorname{ann}_A V$, then we can think of V as having been lifted from an A/J-module. In particular, V is faithful if and only if it is not lifted from any proper homomorphic image of A. Similarly, if A is a b-algebra, we might say that V is b-faithful if it is not lifted from any proper b-algebra homomorphic image of A. In other words, V is b-faithful if and only if $\operatorname{ann}_A V$ contains no nonzero b-ideal of A. Thus Corollary 2 asserts that any b-faithful module V gives rise to the faithful tensor module $\mathcal{T}(V)$. This is essentially Burnside's Theorem.

Let us look at some examples. To start with, recall that a multiplicative semigroup G is a set having an associative multiplication and an identity element 1. Semigroups may contain a zero element $0 \neq 1$ satisfying $0g = g0 = 0$ for all $g \in G$, and as usual we let $G^{\#} = G \setminus \{0\}$ denote the set of nonzero elements of G. The semigroup algebra $K[G]$ is then a K-vector space with basis $G^{\#}$ and with multiplication inherited from that of G. Notice that the zero element of G, if it exists, is identified with the zero element of $K[G]$. Furthermore, $K[G]$ is a b-algebra with Δ defined by $\Delta(g) = g \otimes g$ for all $g \in G^{\#}$. Given this comultiplication, it is easy to see that the only possible bialgebra structure on $K[G]$ would have counit ϵ given by $\epsilon(g) = 1$ for all $g \in G^{\#}$. But then, ϵ is an algebra homomorphism if and only if $G^{\#}$ is multiplicatively closed, or equivalently if and only if there are no zero divisors in G. In other words, most semigroup algebras are just not bialgebras in this way.

If H is also a semigroup, then a semigroup homomorphism $\theta: G \to H$ preserves the multiplication and, by definition, it satifies $\theta(1) = 1$ and $\theta(0) = 0$ if G has a zero element. In particular, it follows that θ extends to a K-algebra homomorphism $\bar{\theta}: K[G] \to K[H]$ which is clearly a b-algebra map. Hence $\ker \bar{\theta}$ is a b-ideal of $K[G]$. As is well known, these are the only possible b-ideals. Since the argument is so simple, we briefly sketch it here.

Let I be a b-ideal of $K[G]$ and let $\bar{\phi}$ be the b-algebra epimorphism defined by $\bar{\phi}: K[G] \to K[G]/I = C$. Then $H = \bar{\phi}(G)$ is a multiplicative subsemigroup of C and, since $\bar{\phi}$ is a b-algebra homomorphism, it is easy to see that H consists of group-like elements. In particular, it follows from [Sw, Proposition 3.2.1(b)] that $H^{\#}$ is a linearly independent subset of C. Furthermore, since $G^{\#}$ spans $K[G]$, we know that $H^{\#}$ spans C. Thus it is clear that $C = K[H]$ and that the map $\bar{\phi}: K[G] \to K[H]$ is the natural extension of the semigroup epimorphism $\phi: G \to H$, namely, the restriction of $\bar{\phi}$ to G. Since $I = \ker \bar{\phi}$, this fact is proved.

By combining the above with Corollary 2, we can quickly obtain Steinberg's generalization of the classical result of Burnside [B, §226]. The original Burnside theorem concerned modules for the complex group algebra $\mathbb{C}[G]$ with $|G| < \infty$, and the proof used the character theory of finite groups. The argument in [St] is more transparent and, of course, it is more general. But the following proof, due to Rieffel in [Ri], shows precisely why the G-faithfulness assumption on the $K[G]$-module V is both natural and relevant.

Let G be a semigroup and let V be a $K[G]$-module. We say that G acts faithfully on V if for all distinct $g_1, g_2 \in G$ we have $(g_1 - g_2)V \neq 0$. Of course, if G is a group, then this condition is equivalent to $(g-1)V \neq 0$ for all $1 \neq g \in G$.

Theorem 3. *Let G be a semigroup and let G act faithfully on the $K[G]$-module V. Then $K[G]$ acts faithfully on the tensor module $\mathcal{T}(V) = \oplus\sum_{n=1}^{\infty} V^{\otimes n}$.*

Proof. Let I be a b-ideal of $K[G]$ contained in $\operatorname{ann}_{K[G]} V$. As we observed, there exists a semigroup epimorphism $\phi: G \to H$ such that I is the kernel of the corresponding algebra map $\bar{\phi}: K[G] \to K[H]$. If $I \neq 0$, then ϕ cannot be one-to-one on G and hence there exist distinct $g_1, g_2 \in G$ with $\bar{\phi}(g_1 - g_2) = 0$. In particular, this implies that $g_1 - g_2 \in I$, so $(g_1 - g_2)V = 0$, contradicting the fact that G is faithful on V. In other words, the G-faithfulness assumption implies that $\operatorname{ann}_{K[G]} V$ contains no nonzero b-ideal. Corollary 2 now yields the result. □

An analogous result holds for enveloping algebras. For simplicity of notation, let us assume that either

(1) K is a field of characteristic 0, L is a Lie algebra over K, and $U(L)$ is its enveloping algebra, or

(2) K has characteristic $p > 0$, L is a restricted Lie algebra over K, and $U(L)$ is its restricted enveloping algebra.

In either case, $U(L)$ is a b-algebra, and in fact a Hopf algebra, with comultiplication determined by $\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$ for all $\ell \in L$. Furthermore, if H is a second (restricted) Lie algebra and if $\theta: L \to H$ is a (restricted) Lie algebra homomorphism, then θ extends uniquely to a b-algebra homomorphism $\bar{\theta}: U(L) \to U(H)$. In particular, $\ker \bar{\theta}$ is a b-ideal of $U(L)$. As is well known, the converse is also true, namely, every b-ideal of $U(L)$ arises in this manner. The argument for this is elementary and similar to the one for semigroup rings. A sketch of the proof is as follows.

Let I be a b-ideal of $U(L)$ and let $\bar{\phi}$ be the b-algebra epimorphism defined by $\bar{\phi}: U(L) \to U(L)/I = C$. Then $H = \bar{\phi}(L)$ is a (restricted) Lie subalgebra of C and H generates C as a K-algebra. In particular, if $\{ h_i \mid i \in \mathscr{I} \}$ is a basis for H, indexed by the ordered set $(\mathscr{I}, \prec)$, then C is spanned by monomials of the form $h_{i_1}^{e_1} h_{i_2}^{e_2} \cdots h_{i_n}^{e_n}$ with $i_1 \prec i_2 \prec \cdots \prec i_n$ and with integers $e_j \geq 0$. Furthermore, when $\operatorname{char} K = p > 0$ and L is restricted, then $e_j < p$ for all j. Since $\bar{\phi}$ is a b-algebra epimorphism, it follows that the elements of H are primitive. Thus, by the work of [Sw, Chapter 13], these straightened monomials are K-linearly independent and therefore $C = U(H)$. In other words, the map $\bar{\phi}: U(L) \to U(H)$ is the natural extension of the (restricted) Lie algebra epimorphism $\phi: L \to H$ where, of course, ϕ is the restriction of $\bar{\phi}$ to L. Since $I = \ker \bar{\phi}$, this fact is proved.

Now let V be a $U(L)$-module. We say that L acts faithfully on V if, for all $0 \neq \ell \in L$, we have $\ell V \neq 0$. The Lie algebra analog of the preceding result is then

Theorem 4. *Let $U(L)$ be a (restricted) enveloping algebra satisfying (1) or (2) above. If L acts faithfully on the $U(L)$-module V, then $U(L)$ acts faithfully on the tensor module $\mathscr{T}(V) = \oplus\sum_{n=1}^{\infty} V^{\otimes n}$.*

As indicated in [M], a theorem of this nature can be used to prove the following interesting result of Harish-Chandra [H, Theorem 1]. Recall that a K-algebra A is residually finite if the collection of its ideals I of finite codimension has intersection equal to 0. In other words, these algebras are precisely the subdirect products of finite-dimensional K-algebras.

Corollary 5. *If L is a finite-dimensional Lie algebra over a field K of characteristic 0, then $U(L)$ is residually finite.*

Proof. By Ado's theorem (see [J, §VI.2]), $A = U(L)$ has a finite K-dimensional module V on which L acts faithfully. Thus, the preceding theorem implies that $0 = \operatorname{ann}_A \mathscr{T}(V) = \bigcap_{n=1}^{\infty} I_n$, where $I_n = \operatorname{ann}_A V^{\otimes n}$. But each $V^{\otimes n}$ is a finite-dimensional A-module, so $I_n = \operatorname{ann}_A V^{\otimes n}$ is an ideal of A of finite codimension, and the result follows. □

If L is a finite-dimensional restricted Lie algebra, then its restricted enveloping algebra $U(L)$ is also finite dimensional. Thus the characteristic $p > 0$ analog of the above is trivial. On the other hand, infinite-dimensional analogs in all characteristics are obtained in [M].

In the remainder of this paper we will restrict our attention to finite-dimensional Hopf algebras. To start with, a Hopf algebra (A, Δ, ϵ, S) is a bialgebra with antipode $S: A \to A$, and a Hopf ideal is the kernel of a Hopf algebra homomorphism. It is easy to see that $I \triangleleft A$ is a Hopf ideal if and only if it is a b-ideal with $\epsilon(I) = 0$ and $S(I) \subseteq I$. Similarly, a K-subalgebra B of A is a Hopf subalgebra if and only if it is a b-subalgebra which is closed under the antipode S. Of course, the b-subalgebra condition means that $\Delta(B) \subseteq B \otimes B$. The following is a special case of a surprising result due to Nichols [N, Theorem 1]. A simple proof of the subalgebra case can also be found in [Ra, Lemma 1].

Lemma 6. *If A is a finite-dimensional Hopf algebra, then any b-subalgebra of A is a Hopf subalgebra and any b-ideal of A is a Hopf ideal of A.*

Proof. Let B denote either a b-subalgebra of A or a b-ideal of A. Furthermore, let $E = \operatorname{Hom}_K(A, A)$ be the convolution algebra of A and set

$$F = \{ f \in E \mid f(B) \subseteq B \}.$$

Certainly F is a K-subspace of E and, in fact, F is closed under convolution multiplication. To see the latter, let $f, g \in F$. If B is a b-subalgebra of A, then $\Delta(B) \subseteq B \otimes B$ implies that

$$(f*g)(B) \subseteq f(B)g(B) \subseteq B^2 = B.$$

On the other hand, if B is a b-ideal of A, then $\Delta(B) \subseteq A \otimes B + B \otimes A$ implies that

$$(f*g)(B) \subseteq f(A)g(B) + f(B)g(A) \subseteq AB + BA = B$$

since $B \triangleleft A$.

Now observe that the identity map Id is contained in F. Thus, by the above, F contains the convolution powers Id^{*n} of Id for all $n > 0$. Furthermore, since A is finite dimensional, E is also finite dimensional and hence the map Id is algebraic over K. In particular, for some $m \geq 1$, we can write Id^{*m} as a finite K-linear combination of the powers Id^{*i} with $i > m$. But Id has convolution inverse S, so by multiplying the expression for Id^{*m} by S^{*m} and by $S^{*(m+1)}$ in turn, we deduce first that $\epsilon = \mathrm{Id}^{*0} \in F$ and then that $S = \mathrm{Id}^{*(-1)} \in F$. In other words, $\epsilon(B) \subseteq B$ and $S(B) \subseteq B$.

Finally, if B is a b-subalgebra of A, then $S(B) \subseteq B$ implies that B is a Hopf subalgebra. On the other hand, if B is a b-ideal of A, then $\epsilon(B) \subseteq B \cap K = 0$. Thus, since $S(B) \subseteq B$, we conclude that B is a Hopf ideal of A. □

The preceding result is false in general for infinite-dimensional Hopf algebras. Some rather complicated counterexamples appear in [N].

Theorem 7. *Let A be a finite-dimensional Hopf algebra.*

(i) *If $\mathscr{F}$ is a family of A-modules which is closed under tensor product, then $\bigcap_{V\in\mathscr{F}} \operatorname{ann}_A V$ is a Hopf ideal of A.*
(ii) *Suppose V is an A-module whose annihilator contains no nonzero Hopf ideal of A. Then $\mathscr{T}(V) = \oplus\sum_{n=1}^{\infty} V^{\otimes n}$ is a faithful A-module.*

This follows immediately from Proposition 1, Corollary 2, and Lemma 6. We can now obtain some consequences of interest. First, recall that an A-module V is semisimple if it is a direct sum of simple modules.

Corollary 8. *If A is a finite-dimensional Hopf algebra, then the set of semisimple A-modules is closed under tensor product if and only if the Jacobson radical $J(A)$ is a Hopf ideal of A.*

Proof. Let $\mathscr{F}$ be the set of all semisimple A-modules. If $\mathscr{F}$ is closed under $\otimes$, then Theorem 7(i) implies that $J(A) = \bigcap_{V\in\mathscr{F}} \operatorname{ann}_A V$ is a Hopf ideal of A. Conversely, if $J(A)$ is a Hopf ideal, then $\mathscr{F}$ consists of all the modules for the Hopf algebra $A/J(A)$ and therefore $\mathscr{F}$ is surely closed under tensor product. □

In a similar manner, we prove

Corollary 9. *Let A be a finite-dimensional semisimple Hopf algebra and let $\mathscr{S}$ be a family of simple A-modules. Suppose that, for all $V, W \in \mathscr{S}$, every irreducible submodule of $V \otimes W$ is contained in $\mathscr{S}$. Then $I = \bigcap_{V\in\mathscr{S}} \operatorname{ann}_A V$ is a Hopf ideal of A and $\mathscr{S}$ is the set of all simple A/I-modules.*

Proof. Let $\mathscr{F}$ be the set of all finite direct sums (with multiplicities) of elements of $\mathscr{S}$. Since A is semisimple, the hypothesis implies that $\mathscr{F}$ is closed under tensor product. Hence, by Theorem 7(i), $I = \bigcap_{V\in\mathscr{S}} \operatorname{ann}_A V = \bigcap_{W\in\mathscr{F}} \operatorname{ann}_A W$ is a Hopf ideal of A. Furthermore, since A/I is semisimple, it follows that $\mathscr{S}$ must be the set of all simple A/I-modules. □

Our final consequence uses the fact that any finite-dimensional Hopf algebra A is a Fròbenius algebra [LS, §5] and hence that every simple A-module is isomorphic to a minimal left ideal of A.

Corollary 10. *Let A be a finite-dimensional Hopf algebra and let V be an A-module whose annihilator contains no nonzero Hopf ideal of A. Then every simple A-module is isomorphic to a submodule of $V^{\otimes n}$ for some $n \geq 1$.*

Proof. It follows from Theorem 7(ii) that $\mathscr{T}(V) = \oplus\sum_{n=1}^{\infty} V^{\otimes n}$ is a faithful A-module. Now let W be a simple A-module, so that W is isomorphic to a minimal left ideal $L \subseteq A$. Since $L \neq 0$, we have $L\mathscr{T}(V) \neq 0$ and hence $LV^{\otimes n} \neq 0$ for some $n \geq 1$. In particular, there exists $u \in V^{\otimes n}$ with $Lu \neq 0$. But then the minimality of L implies that $W \cong L \cong Lu \subseteq V^{\otimes n}$, as required. □

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