

Integer Rules with Arithmetic Properties

– Algebraic Reasoning Approach without Arithmetic Models and Interpretations

Arithmetic Properties “for Whole Numbers” are

- Commutative Property of $+$ and \times
- Associative Property of $+$ and \times
- Identity Property of $+$ and \times
- Distributive Property of \times **over** $+$.

We want to extend the number system from whole numbers to integers while the whole number arithmetic is still true for all integers (Remember the whole number set is a subset of integer set!). If we want the usual properties of arithmetic to be true for integers as well, then the following rules must hold. This also means that once we **accept** the fact that the arithmetic properties are still true for integers, we can prove these rules. Thus, integer arithmetic ($+$, $-$, \times , \div with integers) can be developed by the path

Arithmetic Properties for whole numbers (\longrightarrow Arithmetic Properties for integers)
 \longrightarrow Rules \longrightarrow Familiar integer arithmetic.

Now we need to add **one more property for integers** that is essentially the algebraic definition of negative numbers.

- Inverse Property of $+$ (Additive Inverse Property) : For each integer a , there exists a unique integer called *the opposite of a* , denoted by $-a$, which satisfies

$$a + (-a) = 0$$

Now, let’s **explain, justify, prove** the following rules using the above arithmetic properties. This means you should **explain, justify, prove** why the left hand side and the right hand side of each equal sign are the same using the arithmetic properties.

Rules for Integers

Rule 1. $-(-a) = a$ Negative-Negative Rule

[CCSS.Math.Content.6.NS.C.6a] Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite.

$$\begin{aligned} -(-a) &= -(-a) + 0 \\ &= -(-a) + [a + (-a)] \\ &= -(-a) + [(-a) + a] \\ &= [-(-a) + (-a)] + a \\ &= 0 + a \\ &= a \end{aligned}$$

Question for Rule 1: Recall the “colored chip model” for the illustration of $4 - (-3) = 7$, and find the implication of this proof in the model.

Definition of Subtraction $a - b = a + (-b)$

[CCSS.Math.Content.7.NS.A.1c] Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.

Question for Definition: Illustrate the order of the extensions of number system described in CCSSM. What is the main difference between the terms ‘fraction’ and ‘rational number’ used in CCSSM?

[CCSS.Math.Content.3.NF.A.1] Understand a fraction $\frac{1}{b}$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction $\frac{a}{b}$ as the quantity formed by a parts of size $\frac{1}{b}$.

[CCSS.Math.Content.3.NF.A.2] Understand a fraction as a number on the number line; represent fractions on a number line diagram.

[CCSS.Math.Content.6.NS.C.5] Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

[CCSS.Math.Content.6.NS.C.6] Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

whole numbers \rightarrow a fraction (positive only) as a representation \rightarrow a fraction (positive only) as a number on the number line \rightarrow positive and negative numbers (negative integers as the opposites of whole numbers, and negative fractions as the opposites of positive fractions) \rightarrow rational numbers

Rule 2. $a - (-b) = a + b$ Minus-Negative Rule

[CCSS.Math.Content.7.NS.A.1d] Apply properties of operations as strategies to add and subtract rational numbers.

$$\begin{aligned} a - (-b) &= a + (-(-b)) \\ &= a + [(-(-b)) + 0] \\ &= a + [(-(-b)) + [b + (-b)]] \\ &= a + [(-(-b)) + [(-b) + b]] \\ &= a + [[-(-b) + (-b)] + b] \\ &= a + [0 + b] \\ &= a + b \end{aligned}$$

Question for Rule 2: Use **Rule 1** to simplify the steps. Which proofs are more accurate and rigorous do you think? Why? If you think both are at the same level of accuracy or rigor, why do you think so?

Now we will use the arithmetic properties, Rule 1, and Rule 2 to **explain, justify, prove** the the rest of the rules.

Rule 3. $(-a) \times b = -(a \times b)$ Negative-Number-of-Groups Rule

[CCSS.Math.Content.7.NS.A.2] Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.

$$\begin{aligned}
 (-a) \cdot b &= (-a) \cdot b + 0 \\
 &= (-a) \cdot b + [a \cdot b + (-(a \cdot b))] \\
 &= [(-a) \cdot b + a \cdot b] + (-(a \cdot b)) \\
 &= [(-a) + a] \cdot b + (-(a \cdot b)) \\
 &= 0 \cdot b + (-(a \cdot b)) \\
 &= 0 + (-(a \cdot b)) \\
 &= -(a \cdot b)
 \end{aligned}$$

Question for Rule 3: Is $(-a)$ a negative number? Why, or why not? What about b ? What if we assume like that? Do we get into any trouble?

Rule 4. $(-a) \times (-b) = ab$ Negative-times-Negative Rule

[CCSS.Math.Content.7.NS.A.2a] Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

$$\begin{aligned}
 (-a) \cdot (-b) &= (-a) \cdot (-b) + 0 \\
 &= (-a) \cdot (-b) + [a \cdot (-b) + (-(a \cdot (-b)))] \\
 &= [(-a) \cdot (-b) + a \cdot (-b)] + (-(a \cdot (-b))) \\
 &= [(-a) + a] \cdot (-b) + (-(a \cdot (-b))) \\
 &= 0 \cdot (-b) + (-(a \cdot (-b))) \\
 &= 0 + (-(a \cdot (-b))) \\
 &= -(a \cdot (-b)) \\
 &= (-a) \cdot (-b)
 \end{aligned}$$

Question 1 for Rule 4: Find the error in the above proof. What was wrong?

$$\begin{aligned}
(-a) \cdot (-b) &= (-a) \cdot (-b) + 0 \\
&= (-a) \cdot (-b) + [a \cdot (-b) + (-(a \cdot (-b)))] \\
&= [(-a) \cdot (-b) + a \cdot (-b)] + (-(a \cdot (-b))) \\
&= [(-a) + a] \cdot (-b) + (-(a \cdot (-b))) \\
&= 0 \cdot (-b) + (-(a \cdot (-b))) \\
&= 0 + (-(a \cdot (-b))) \\
&= -(a \cdot (-b)) \\
&= -((-b) \cdot a) \\
&= -(-(b \cdot a)) \\
&= b \cdot a \\
&= a \cdot b
\end{aligned}$$

Question 2 for Rule 4: How can you explain this proof to middle school students? What is the core idea in the proof above? $3 \times (-4) = -12$, $(-3) \times (-4) + 3 \times (-4) = (-3+3) \times (-4) = 0 \times (-4) = 0$

Rule 5. $(-a) \div (-b) = a \div b$

Rule 6. $-(a \div b) = (-a) \div b = a \div (-b)$

[CCSS.Math.Content.7.NS.A.2b] Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $(p/q) = (p)/q = p/(q)$. Interpret quotients of rational numbers by describing real-world contexts.

Applications

1. Prove Rule 5, and Rule 6 for Integers using Arithmetic Properties for addition and multiplication and rules that you proved already.
2. Prove $a \times 0 = 0$ using Arithmetic Properties for addition and multiplication and rules that you proved already. Start with $a(1 + 0) = a$.
3. Prove

(a) $-a = (-1) \times a$,

(b) $(a - b) - (c - d) = a - b - c + d$

using the definition of subtraction, $a \times 0 = 0$, and arithmetic properties for addition and multiplication.