Applications of nilpotent Artin-Schreier theory to local counting problems and p-ranks of curves

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Abstract

Abrashkin's nilpotent Artin-Schreier theory describes a way of parametrizing G-extensions of local function fields when G is a p-group of nilpotency class $< p$. We investigate applications of this theory to some local counting problems and the study of p-ranks of wildly ramified cover.

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1 Introduction

Over a characteristic zero local non-Archimedean local field such as $F = \mathbb{Q}_p$, Krasner's lemma implies that the number of extensions of degree n is finite. Krasner's lemma roughly says that nearby polynomials define the same splitting field. The set of Eisenstein polynomials of degree n is homeomorphic to $(\mathfrak{p}\mathcal{O}_F)^n$, a compactness argument guarantees the finiteness of totally ramified extensions of degree n and this is enough to show the general case. Krasner[Kra66] was even able to write an explicit formula for the number of degree n extensions of p-adic fields. This argument follows through in the characterstic p lcoal fields $\mathbb{F}_q(\!(t)\!)$ except in the case when the degree of extension is divisible by the characteristic because of inseparable polynomials. As a result, the question of determining the distribution of number of degree n extensions of a local function field is a rich problem in the wild case.

We will focus on the case of estimating the distribution of G-extensions of a local function field where G is a group whose order is divisible by the characterstic p. In [GS25], the distribution of these G-extensions was studied for an invariant called the last jump and for odd prime-powered p-groups of nilpotency class 2.

1.1 Prior work

For the discriminant, Lagemann[Lag12][Lag15] studied the distribution of discriminants for Artin-Schreier extension and Artin-Schreier-Witt extensions of global and local function fields. Potthast[Pot26] settled the elementary abelian case by providing precise asymptotics for the local and global cases. Forthcoming paper of Bhatanger, Potthast and Yin extends this to abelian p-groups. These asymptotics truly differ from the tame case; the a-constant appearing in the asymptotics of the distribution of discriminants in the global function fields are known to be higher than those predicted by Malle's conjecture.

For the last jump, works of Klüners and Müller [KM20] proved precise asymptotics for abelian p-groups. Gundlach and Seguin's work [GS25] studied the last jump distribution for global and local function fields.

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2 Background Material

For those familiar with Abrashkin's nilpotent Artin-Schreier theory, they may want to skip section §2 and §3.

2.1 Parametrizing extensions

This subsection is devoted to understanding parametrizing étale G-extensions of an arbitrary field as outlined in [GS25]. Let $\Gamma_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ be the absolute Galois group. Let $G_{F^{\mathrm{sep}}}$ be a topological group with a continuous Γ_F-action equipped with a Γ_F-equivariant group homomorphism

$$\sigma : G_{F^{\mathrm{sep}}} \to G_{F^{\mathrm{sep}}}.$$

$G := \{g \in G_{F^{\mathrm{sep}}} : \sigma(g) = g\}$ (the σ-invariants of $G_{F^{\mathrm{sep}}}$).

$G_F := \{g \in G_{F^{\mathrm{sep}}} : \Gamma_F(g) = g\}$ (the Galois invariants of $G_{F^{\mathrm{sep}}}$).

Example (Classical Artin-Schreier Theory)**.** Let F be a field of characteristic p. (For example a function field $F = \mathbb{F}_q(t)$). Let Γ_F act on $G_{F^{\mathrm{sep}}} := F^{\mathrm{sep}}$. Let $\sigma : G_{F^{\mathrm{sep}}} \to G_{F^{\mathrm{sep}}}$ be the absolute Frobenius map $x \mapsto x^p$. In this case, $G = \mathbb{F}_p$ and $G_F = F$.

Definition (Multiplicative Artin-Schreier Map)**.** Consider the left $G_{F^{\mathrm{sep}}}$ on $G_{F^{\mathrm{sep}}}$ given by

$$g \cdot m := \sigma(g) m g^{-1}.$$

The multiplicative Artin-Schreier map is $\wp : G_{F^{\mathrm{sep}}} \to G_{F^{\mathrm{sep}}}$ given by $\wp(g) = g \cdot 1 = \sigma(g) g^{-1}$.

Remark. This left action is almost like conjugating m by g except we twist it by σ when we premultiply. I will call it the twisted conjugation action.

We will write $G_F//_{G_F}$ for the set of orbits of G_F acting on itself via this action.

This is the main proposition regarding the parametrization of G-extensions of F.

Proposition 1. [GS25, Proposition 2.4]

If the following holds,

1. $G \subset G_F$
2. $G_F \subset \wp(G_{F^{\mathrm{sep}}})$
3. $H^1(\Gamma_F, G) \to H^1(\Gamma_F, G_{F^{\mathrm{sep}}})$ is trivial.

then, there is a bijection

$$\begin{aligned} \mathrm{orb} : H^1(\Gamma_F, G) &\to G_F//_{G_F}, \\ [\tau \to g^{-1}\tau(g)] &\mapsto [\wp(g)]. \end{aligned}$$

Abrashkin's nilpotent Artin-Schreier theory tells us how to get ramification information of $G(L)$-extensions of local fields of p where $G(L)$ is a p-group of nilpotency class $< p$ and L is the correponding Lie algebra.

In particular, the main point of nilpotent Artin-Schreier theory is that $G_{F^{\text{sep}}} = G(L \otimes O(F_{\text{sep}}))$ satisfies the conditions written in Proposition 1 and this parametrization gives a way to read off the ramification filtration. If the p-group in consideration is a group of exponent p, we get a simplification; the corresponding Lie algebra is defined over $\mathbb{F}_p$ and the use of Witt ring and Cohen ring becomes unnecessary. That is to say $G(L \otimes_{\mathbb{Z}_p} O(F_{\text{sep}})) = G(L \otimes_{\mathbb{F}_p} F)$.

3 Abrashkin's Nilpotent Artin-Schreier Theory

3.1 Lazard correspondence

The Lazard correspondence is an important correspondence in understanding Abrashkin's nilpotent Artin-Schreier theory.

Theorem 3.1 (Lazard Correspondence for finite p-groups)**.** There is an isomorphism of categories

$\{\text{finite } \mathbb{Z}_p\text{-Lie algebras of nilpotency class } < p\} \longleftrightarrow \{p\text{-groups of nilpotency class} < p\}$.

Call the functor from left to right as $\mathcal{G}$ and $\mathcal{L}$ be the inverse functor.

The Lazard correspondence stems from the Baker-Campbell-Hausdorff Formula and the denominators of the coefficients of the terms appearing in it. It can be viewed as a p-adic analogue of the more familiar correspondence between real Lie algebras and simply-connected Lie-groups.

Example. Let L be a finite $\mathbb{Z}_p$-algebra of nilpotency class $< p$. Taking two elements $x, y \in L$, $\mathcal{G}(L)$ is the set L equipped with the group operation defined by

$$x \circ y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + ...,$$

inverse of x being $-x$. Taking n-th power in the group corresponds to $\underbrace{x \circ x \circ .... \circ x}_{n \text{ times}} = nx$.

Therefore, L is a $\mathbb{Z}/p^i\mathbb{Z}$-Lie algebra if and only if the exponent of $\mathcal{G}(L)$ is p^i.

3.2 Witt Rings and Cohen Rings

We recall the relevant results about Witt rings from [Ser13]. A ring R of characteristic is called perfect if the absolute Frobenius map $\sigma : R \to R, x \mapsto x^p$ is an isomorphism of rings. A ring R is called a strict p-ring if it is p-adically complete, p-torsion free and R/pR is a perfect ring.

If R is a perfect ring of characteristic p, then $W(R)$ is a strict p-ring with residue ring R.

Example. The most well-known example is the case when $R = \mathbb{F}_{p^n}$, a finite field of characteristic p with q elements. One can check that $K = \mathbb{Q}_p(\zeta_{p^n-1})$ is an unramified extension of $\mathbb{Q}_p$ and $R = \mathcal{O}_K$ is a strict p-ring with residue field $\mathbb{F}_{p^n}$.

In general, it is difficult to compute the ring of Witt vectors of an imperfect ring. However, the main objects of our study are fields like $\mathbb{F}_q(t)$ or $\mathbb{F}_q((t))$ which are imperfect.

One thing we can do is to take the Witt vectors of the perfection of a ring.

3.3 Extensions of local or global fields of characteristic p

In this subsection, we will say how we parametrize extensions of global fields of characteristic p and how it relates to the Abrashkin's nilpotent Artin-Schreier theory.

Proposition 2. [GS25, 2.19] Let F be a local or global field of characteristic p. Let L be a finite Lie algebra over $\mathbb{Z}_p$ of nilpotency class $< p$. Let $\mathcal{G}(L)$ be the corresponding p-group.

Then, there is an isomorphism,

$$\mathrm{orb} : H^1(\Gamma_F, \mathcal{G}(L)) \to G(L \otimes_{\mathbb{Z}_p} W(F^{\mathrm{perf}}))//_{\mathcal{G}(L\otimes_{\mathbb{Z}_p} W(F^{\mathrm{perf}}))}.$$

Remark. Whilst $\mathcal{G}(L\otimes_{\mathbb{Z}_p} W(F^{\mathrm{perf}}))$ suffers from the complication of working over the Witt ring of perfect closure, after taking the orbits of it under the action by itself, it has a simpler description.

Since $e_1 \sim e_2 \iff e_1 = (\sigma c) \circ e_2 \circ (-c)$, one can see that $t^{1/p^{n+1}} \sim t^{1/p^n}$ by taking $e_1 = t^{1/p^n}, e_2 = c = t^{1/p^n}$. Thus, an equivalence class in $\mathcal{G}(L \otimes_{\mathbb{Z}_p} W(F^{\mathrm{perf}}))//_{\mathcal{G}(L\otimes_{\mathbb{Z}_p} W(F^{\mathrm{perf}})}$ can be represented by summand of terms with only integral powers of t. By the same argument of [Abr95, 1.3], we can choose a represenative of the form $\sum_{a\in\mathbb{Z}^0(p)} t^{-a} D_{a0}$ with $D_{a0} \in L \otimes W(\mathbb{F}_q)$, $D_{00} \in \alpha L$ and α is a fixed element of $W(\mathbb{F}_q)$ of trace 1.

This leads to the following description of $\mathcal{G}(L \otimes_{\mathbb{Z}_p} W(F^{\mathrm{perf}}))//_{\mathcal{G}(L\otimes_{\mathbb{Z}_p} W(F^{\mathrm{perf}}))}$.

Theorem 3.2. [GS25, 3.6] There is a bijection,

$$\alpha^0 : L \otimes_{\mathbb{Z}_p} W(F^{\mathrm{perf}}))//_{\mathcal{G}(L\otimes_{\mathbb{Z}_p} W(F^{\mathrm{perf}})} \to L \otimes_{\mathbb{Z}_p} \left(\bigoplus_{a\in\mathbb{N}\setminus p\mathbb{N}\cup\{0\}} W(k)t^{-a} \right) //_{W(k)}.$$

3.4 The ramification filtration

We outline how Abrashkin's nilpotent Artin-Schreier theory gives us information on the ramification filtration of local extensions.

Let $x = \sum_{a\in\mathbb{N}^0(p)} D_a t^{-a} \in L \otimes \mathcal{D}^0$. This corresponds to a map $\psi \in H^1(\Gamma_K, \mathcal{G}(L))$. The image of $\{\psi(\Gamma_K^{(v)})\}_{v\geq 0}$ where $\Gamma_K^{(v)}$ is the ramification group in upper numbering induces a filtration on the Lie algebra which we denote $L(v)$.

Abrashkin gives a description of the Lie algebra $L(v)$ in terms of $\{D_a : a \in \mathbb{N}^0(p)\}$. Before that, we give a definition.

Definition. Let $D_{a,n}$ to be $\sigma^n(D_a)$. Define

$$F_{\gamma,-N} := \sum_{\substack{n_1 \geq 0 \\ n_1 \geq \ldots \geq n_s \geq -N \\ \gamma = a_1 p^{n_1} + a_2 p^{n_2} + \ldots + a_s p^{n_s}}} \eta(n_1, \ldots, n_s) a_1 p^{n_1} [\ldots [D_{a_1,n_1}, D_{a_2,n_2}], , \ldots], D_{a_s,n_s}] \in L \otimes W(\mathbb{F}_q)$$

where $1 \leq s < p$ and for $s = 1$, we take that to mean D_{a_1,n_1}.

This should be the covariant version.

The next theorem is the main result from Abrahskin's nilpotent Artin-Schreier theory that we will be leveraging. See [Abr98, §1.4] for details on the covariant nilpotent Artin-Schreier theory.

Theorem 3.3. [Abr98, §4 Theorem 1] For any $v \geq 0$, there is a natural number $N(v)$ such that if $N \geq N(v)$ then, $L(v)W(\mathbb{F}_q)$ is the Lie algebra generated by $\{\mathcal{F}_{\gamma,-N} : \gamma \geq v\}$.

Remark (Technical Comment on the Bad Pairs)**.** In [GS25], they impose what may appear to be extra conditions on $a_1 p^{n_1}$ and $a_2 p^{n_1}$ (not $a_2 p^{n_2}$) in their definition of $\mathcal{F}_{\gamma,-N}$. For a fixed γ, $\mathcal{F}_{\gamma,-N}$ stabilizes for sufficiently large N (see [Abr98, 2.3] for details on bad pairs). In this formulation by [GS25], they take $\mathcal{F}_{\gamma,-N}$ where the stabilization hasn't happened but still guarantees it generates the same Lie algebra associated to the ramification filtration.

Our formulation of the nilpotent Artin-Schreier theory follows Abrashkin's original setup more closely by not imposing these extra conditions but keeping the dependency on N. For example, when studying integral breaks in the ramification filtration, taking $N = 0$ is sufficient. If all ramification breaks are contained in $\frac{1}{p^k}\mathbb{Z}$, then we can take $N = k$. In general, for a p-group G, the ramification breaks are contained in $\frac{1}{|G|}\mathbb{Z}$ but in many cases, we can get show that the ramification breaks are contained in a strictly smaller subgroup.

4 Some group theory

We will be introducing some of the p-groups that we will be studying and the relevant notions required for the next part.

In the study of p-groups, we will be using the central series very often.

Lower Central Series

$$\{\mathrm{id}\} = \gamma_{m+1}(G) \trianglelefteq \ldots \trianglelefteq \gamma_i(G) \trianglelefteq \ldots \trianglelefteq \gamma_2(G) \trianglelefteq G$$

with $\gamma_{i+1}(G) = [\gamma_i(G), G]$.

Upper Central Series

$$\{\mathrm{id}\} = \zeta_0(G) \trianglelefteq \zeta_1(G) \trianglelefteq ... \trianglelefteq \zeta_{m-1}(G) \trianglelefteq G$$

with $\zeta_{i+1}(G) := \{x \in G : \forall y \in G : [x, y] \in \zeta_i(G)\}$. i.e. $\zeta_{i+1}(G)$ is the maximal subgroup such that $[\zeta_{i+1}(G), G] \subset \zeta_i(G)$.

Definition. A p-group is said to be of maximal class if it is of order p^{n+1} and nilpotency class n.

More generally, given a p-group with nilpotency class n, and if the order of the p-group is p^m, then the coclass c is the difference $m - n$.

Example 1. Consider $G := (\mathbb{F}_p^n) \rtimes \mathbb{F}_p \subset \mathrm{AGL}_n(\mathbb{F}_p)$. The semidirect product structure can be described as follows: $\varphi : \mathbb{F}_p \to \mathrm{Aut}(\mathbb{F}_p^n), 1 \mapsto \mathrm{id} + \sum_{i=1}^{n-1} E_{i,i+1}$. It is not difficult to show that this group has nilpotency class n.

Let g be the element in G that acts as $v \mapsto Av$ where $A = \mathrm{id} + \sum_{i=1}^{n-1} E_{i,i+1}$.

Let h_n be the element in G that acts as $v \mapsto v + e_n$.

One can check that the group commutator $g \circ h_n \circ g^{-1} \circ h_n^{-1}$ is the element that acts as $v \mapsto v + e_{n-1}$. In a similar fashion, let $h_i : v \mapsto v + e_i$ for $1 \leq i \leq n$. Then, $g^{-1}h_i^{-1}gh_i = h_{i-1}$ for $2 \leq i \leq n$ and $g^{-1}h_1^{-1}gh_1 = \mathrm{id}$.

Later, we will see that this family of p-groups has very interesting properties when we look at the degree of commutativity.

Remark. When $n = 2$, $(\mathbb{F}_p)^2 \rtimes \mathbb{F}_p$ is in fact isomorphic to the Heisenberg group over $\mathbb{F}_p$ which is $\mathcal{H}_3(\mathbb{F}_p) := \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{F}_p \right\}$. One can check that $(\mathbb{F}_p^2) \rtimes \mathbb{F}_p \to \mathcal{H}_3(\mathbb{F}_p), (v \mapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} v + \begin{bmatrix} a \\ b \end{bmatrix}) \mapsto \begin{bmatrix} 1 & b & a \\ & 1 & n \\ & & 1 \end{bmatrix}$ is an isomorphism.

Remark. In general, the lower and upper central series need not agree and we only have the containment $\zeta_i(G) \subset \gamma_{m+1-i}(G)$. The length of these central series do agree and m is the nilpotency class of G.

For p-group of maximal class, the two central series do agree and all successive quotients are of order p except for $G/\gamma_2(G)$; this is isomorphic to the non-cyclic group of order p^2.

In the theory of p-groups of maximal class, a very useful concept is that of the two-step centralizer. See [LGM02, §3] for details on proofs.

Definition (Two-step centralizer)**.** For $i = 2, ..., m - 1$, the two-step centralizer is

$$K_i = C_G(\gamma_i(G)/\gamma_{i+2}(G)).$$

These K_i are maximal subgroups of G.

Take the lower central series from before and rewrite G as $\gamma_0(G)$. Let $\gamma_1(G) = K_2$. This gives a filtration with all successive quotients being of order p.

Lemma 4.1. Let G be a p-group of maximal class. There exist elements $g, h \in G$ such that if we let $g_1 = g$ and $g_{i+1} = [g_i, h]$ then $x_n \neq \text{id}$ where n is the nilpotency class of G. Every element of G can be written as a product of g_i and h. In particular, G is generated by g and h.

Sketch proof. Choose $h \in G \setminus \gamma_1(G)$ and $g \in \gamma_1(G) \setminus \gamma_2(G)$. □

Remark. Lemma 4.1 gives us a way of describing p-groups of maximal class. Once we have defined g_i and h, defining the commutators $[g_i, g_j]_{\mathbf{Grp}}$ gives us the group law. One has to define $[g_i, g_j]_{\mathbf{Grp}}$ so that all Hall-Witt identities are satisfied. If one were to similarly define $\mathbb{F}_p$-Lie algebras of cardinality p^{n+1} and nilpotency class n, we can find two elements x, y and L would have an $\mathbb{F}_p$-basis of $\{y, x_1, ..., x_n\}$ with $x_{i+1} := [x_i, y]_{\mathbf{LieAlg}}$. One then needs to define $[x_i, x_j]_{\mathbf{LieAlg}}$ accordingly so that the Jacobi identities are all satisfied.

Lemma 4.2. $[\gamma_i(G), \gamma_j(G)] \subset \gamma_{i+j}(G)$ for all $i, j \geq 1$.

Definition (Degree of Commutativity)**.** The degree of commutativity of a CF-group is defined as

$$[\gamma_i(G), \gamma_j(G)] \subset \gamma_{i+j+d}(G)$$

for all $i, j \geq 1$. If $\gamma_1(G)$ is abelian, the degree of commutativity is $n - 1$ where n is the nilpotency class. (i.e. $[\gamma_1(G), \gamma_1(G)] = \{\text{id}\} = \gamma_{n+1}(G)$.)

Remark. If $\gamma_1(G)$ is abelian, some authors[LGM02] use the convention that the degree of commutativity is $n - 2$.

Example. The group $G = (\mathbb{F}_p)^n \rtimes \mathbb{F}_p$ has an abelian subgroup of index p. This maximal abelian subgroup is in fact, $\gamma_1(G)$ and thus, it has a degree of commutativity $n - 1$.

An important thing we need to discuss is how the group commutators and Lie brackets are related. When there is ambiguity, we will denote $[-, -]_{\mathbf{LieAlg}}$ to mean the Lie bracket whereas, $[-, -]_{\mathbf{Grp}}$ to mean the group commutator.

Lemma 4.3. Let L be a finite $\mathbb{Z}_p$-Lie algebra of nilpotency class $< p$ and $G = \mathcal{G}(L)$. Then,

$$[x, y]_{\mathbf{LieAlg}} \equiv [x, y]_{\mathbf{Grp}} \pmod{\text{degree three commutators}}.$$

Proof. Since $x \circ y \equiv x + y + \frac{1}{2}[x, y]$ (mod degree three commutators) and $(-x) \circ (-y) \equiv -x - y + \frac{1}{2}[x, y]$ (mod degree three commutators), taking the group composition $(x \circ y) \circ ((-x) \circ (-y))$ yields the desired relation. □

By inducting on this lemma, it is not hard to see the following corollary.

Corollary 4.1. $[x_1, [x_2, ...[x_{i-1}, x_i]]]_{\mathbf{LieAlg}} \equiv [x_1, [x_2, ...[x_{i-1}, x_i]]]_{\mathbf{Grp}}$ (mod degree i+1 commutators).

Proof. The base case has already been covered. Suppose the inductive hypothesis is true for $i-1$, $[x_2, ...[x_{i-1}, x_i]]_{\mathbf{LieAlg}} = [x_2, ..., [x_{i-1}, x_i]]_{\mathbf{Grp}} + E_{i-1}$ where E_j is a sum of terms of commutators of weight degree at least j.

Then, taking the commutator with x_1 gives us

$[x_1, [x_2, ...[x_{i-1}, x_i]]_{\mathbf{LieAlg}}]_{\mathbf{Grp}} = [x_1, [x_2, ..., [x_{i-1}, x_i]]]_{\mathbf{LieAlg}} + [x, E_{i-1}]_{\mathbf{Grp}}$

We now analyze the two two terms, $[x_1, [x_2, ...[x_{i-1}, x_i]]_{\mathbf{LieAlg}}]_{\mathbf{Grp}}$ and $[x, E_{i-1}]_{\mathbf{Grp}}$ using our inductive hypothesis.

1. $[x_1, [x_2, ...[x_{i-1}, x_i]]_{\mathbf{LieAlg}}]_{\mathbf{Grp}} = [x_1, [x_2, ...[x_{i-1}, x_i]]]_{\mathbf{LieAlg}} + E$

   where E is a sum of degree ≥ 3 commutators involving x_1 and $[x_2, ...[x_{i-1}, x_i]]$. In particular, it is a sum of degree $\geq i+1$ commutators involving $x_1, x_2, ..., x_i$.

2. $[x, E_{i-1}]_{\mathbf{Grp}} = [x, E_{i-1}]_{\mathbf{LieAlg}}$+(degree 3 commutators involving x, E_{i-1}). Since commutators involving x and E_i are themselves commutator of degree $i+1$, we can absorb this into the error term.

This concludes the inductive step. □

Remark. We now remark that the notions of lower central series, two-step centralizer, maximal class, coclass and degree of commutativity can all be defined for the $\mathbb{Z}_p$-Lie algebras we will be studying. For instance, $\gamma_{i+1}(L) := [L, \gamma_i(L)]$ for $i \geq 2$ and $\gamma_1(L) := \{x \in L : [x\gamma_4(L), y\gamma_4(L)] = 0, \forall y \in \gamma_2(L)\}$.

Corollary 4.2. $\mathcal{G}(\gamma_i(L)) = \gamma_i(\mathcal{G}(L))$.

Remark. We earlier remarked that $G = (\mathbb{F}_p)^n \rtimes \mathbb{F}_p$ can be generated by two elements, g, h with $g_1 = g$ and $g_{i+1} = [g_i, h]_{\mathbf{Grp}}$. The relations were $[g_i, g_j]_{\mathbf{Grp}} = \mathrm{id}$ whenever $i, j \geq 1$. On the other hand, we can consider the $\mathbb{F}_p$-Lie algebra $L = y\mathbb{F}_p \oplus (\oplus_{i=1}^n x_i\mathbb{F}_p)$ generated by two elements x, y with $x_1 = x$ and $x_{i+1} = [x_i, y]_{\mathbf{LieAlg}}$ with relations $[x_i, x_j]_{\mathbf{LieAlg}} = 0$.

Then, there exists a unipotent matrix that sends the ordered basis $\{y, x_1, x_2, ..., x_n\}$ to $\{h, g_1, ..., g_n\}$ so that it agrees with the isomorphism (of Lie-algebras), $L \to \mathcal{L}(G)$.

5 Main Results for p-groups of maximal class

In this section, we collect the main results on counting the number of G-extensions of $\mathbb{F}_q(\!(t)\!)$ with bounded last jump. The particular family of p-groups we will be looking at is the p-groups of maximal class and their mild generalizations (i.e. CF-groups and ECF-groups).

Definition. Let G be a p-group of nilpotency class $< p$ and $F = \mathbb{F}_q(\!(t)\!)$ where $\mathrm{char}(F) = p$. Define $Z(F, G, M) = \#\{L/F : \mathrm{Gal}(L/F) \cong G, \mathrm{lastjump}(L/F) \leq M\}$.

Our strategy is as follows. We will construct certain parametrizing subsets of G-extensions of F with a fixed last jump. That is to say every element in this subset should have the same last jump. The goal is to construct large sets with small last jumps and this should give us good lower bounds on $Z(F, G, M)$.

The terms in Abrashkin's $\mathcal{F}_{\gamma,-N}$ is affected by how we take Lie brackets of elements of L and the exponent of these Lie brackets. We will be often making the assumption that the group G has exponent p when we are studying certain families of p-groups for simplification. When the group G does not have exponent p, it will be useful to know about the subgroups with exponent p.

How far we can push these lower bounds will depend on constructing subgroups of low-index with desired properties such as having low nilpotency class. There will be isomorphism class of p-groups in many of the families that we look at, that have much better lower bounds. As a rule of thumb, the closer G is to being abelian (e.g. higher degree of commutativity, G having low-index abelian subgroups), the more likely $Z(F, G, M)$ is to be higher.

5.1 Main counting result for CF-groups and p-groups of maximal class with exponent p

We will study a particular family of p-groups which also contain the p-groups of maximal class.

Definition (CF-groups)**.** A CF-group is a p-group of order p^n such that $\gamma_i(G)/\gamma_{i+1}(G) \cong C_p$ for all $i = 2, ..., m-1$ where $m < n$ is the nilpotency class. If $m = n-1$, then G is of maximal class. A CF-group is said to be an ECF-group if $G/\gamma_2(G)$ is elementary abelian.

Theorem 5.1. Let G be a p-group that is a CF-group of exponent p, nilpotency class $n < p$ and coclass c. Then,

$$Z(F, G; N) \gg_{q,n,d,c} q^{N(1-1/p)\left(An+\frac{c}{n}\right)}$$

where $A = \pi^2/6 - 1 > 0.6449$

Proof. Let $H_a := \gamma_{\lceil \frac{(n+1)}{a+1} \rceil}(G)$. It is not difficult to see that $[\underbrace{H_a, ..., H_a}_{a+1}] = \{\mathrm{id}\}$ or in other words, H_a has nilpotency class $\leq a$ and because G is a CF-group, H_a has order $p^{\lfloor \frac{a(n+1)}{a+1} \rfloor}$.

Let N be any positive integer coprime to p and consider,

$$X_N := \left\{ \sum_{i \in \mathbb{Z}_0(p) \cap [0,N]} D_i t^{-i} : D_N \neq 0, D_i \in \begin{cases} L(H_a) \otimes \mathbb{F}_q & N/(a+1) < i \leq N/a \text{ for all } a = 1, ..., n-1 \\ L(G) \otimes \mathbb{F}_q & 0 \leq i \leq N/n \end{cases} \right\}$$

Claim. Every $x \in X_N$ has a last jump of N.

We estimate the cardinality of X_N.

$$|X_N| = \left(\prod_{a=1}^{n-1} |L(H_a) \otimes \mathbb{F}_q|^{\frac{N}{a(a+1)}(1-1/p)+O(1)} \right) \cdot |L(G) \otimes \mathbb{F}_q|^{N/n(1-1/p)+O(1)}$$

$$|X_N| \gg_{q,n,d,c} \left(\prod_{a=1}^{n-1} q^{\lfloor \frac{a(n+1)}{a+1} \rfloor \frac{N}{a(a+1)}(1-1/p)} \right) \cdot q^{(n+c)\frac{N}{n}(1-1/p)}$$

We may break the sum as follows,

$$\sum_{a=1}^{n-1} \left\lfloor \frac{a(n+1)}{a+1} \right\rfloor \frac{1}{a(a+1)} = \sum_{a=1}^{n-1} \left(n+1 - \left\lceil \frac{n+1}{a+1} \right\rceil \right) \frac{1}{a(a+1)} = (n+1)\left(1 - \frac{1}{n}\right) - \sum_{a=1}^{n-1} \frac{1}{a(a+1)} \left\lceil \frac{n+1}{a+1} \right\rceil.$$

We may now lower bound this as,

$$\geq \left(n - \frac{1}{n}\right) - \sum_{a=1}^{n-1} \frac{1}{a(a+1)} \frac{(n+1)+a}{a+1} = \left(n - \frac{1}{n}\right) + \sum_{a=1}^{n-1} \frac{n}{(a+1)^2} - \sum_{a=1}^{n-1} \frac{n+1}{a(a+1)} \geq n(\pi^2/6 - 1 - 1/n)$$

$$Z(F, G; N) \gg_{q,n,d,c} q^{N(1-1/p)\left(An + \frac{c}{n}\right)}$$

where $A = \pi^2/6 - 1 > 0.6449$ □

Example. Suppose G is a p-group of maximal class and exponent p. Then, we have $Z(F, G; N) \gg q^{N(1-1/p)(0.6449n)}$. In many cases, with further knowledge of other properties of the group, we can get better lower bounds.

We now state a version where we have knowledge of the degree of commutatitivity. We will split this into two cases where the degree of commutativity is high and when the degree of commutativity is low.

Theorem 5.2 (High Degree of Commutativity)**.** Let G be a p-group that is a CF-group of exponent p, nilpotency class n, coclass c and degree of commutativity $d \geq (n-1)/2$. Then,

$$Z(F, G; N) \gg q^{(\frac{3n+d+2c-2}{4})N(1-1/p)}$$

If $d = n - 1$, then we may further say that

$$Z(F, G, ; N) \gg q^{(n+c-1)N(1-1/p)}.$$

Proof. Let $H_a = \gamma_{f(a)}(G)$ where $f(a) = \max\{1, \lceil \frac{(n+1)-ad}{a+1} \rceil\}$.
Again, $[\underbrace{H_a, ..., H_a}_{a+1}] = \{\mathrm{id}\}$.
Let N be any positive integer coprime to p and consider,

$$X_N := \left\{ \sum_{i \in \mathbb{Z}_0(p) \cap [0,N]} D_i t^{-i} : D_N \neq 0, D_i \in \begin{cases} L(H_1) \otimes \mathbb{F}_q, & \text{for } N/2 < i \leq N \\ L(\gamma_1(G)) \otimes \mathbb{F}_q, & \text{for } 0 \leq i \leq N/2 \end{cases} \right\}$$

Now estimating the cardinality of X_N, we see that
$|X_N| = |L(H_1) \otimes \mathbb{F}_q|^{\frac{N}{2}(1-1/p)+O(1)} \cdot |L(\gamma_1(G) \otimes \mathbb{F}_q|^{\frac{N}{2}(1-1/p)+O(1)}$
Thus, $|X_N| \gg (q^{\lfloor \frac{n+d+1}{2} \rfloor})^{\frac{N}{2}(1-1/p)} \cdot (q^{n+c-1})^{\frac{N}{2}(1-1/p)}$
Hence, $Z(F, G; N) \gg q^{N\frac{3n+d+2c-2}{4}(1-1/p)}$

□

Example. If $G = (\mathbb{F}_p)^n \rtimes \mathbb{F}_p$ $(n \geq 2)$, it is a p-group of maximal class and of degree of commutativity $n-1$. Thus, $Z(F, G; N) \gg q^{nN(1-1/p)}$.

Remark. Since the degree of commutativity d is a uniform statement on how the commutator $[\gamma_i(G), \gamma_j(G)]$ interacts with the subgroup in the lower central series, it is pretty common for $[\gamma_i(G), \gamma_j(G)] \subsetneq \gamma_{i+j+d}(G)$ to hold. Put it differently, there may only be a few commutator relations that is being a bottleneck for the degree of commutativity to be higher. For instance, without any assumption on the degree of commutativity, $[\gamma_1(G), \gamma_1(G)] \subset \gamma_4(G)$ always holds which follows from the definition of $\gamma_1(G)$ as $C_G(\gamma_2(G)/\gamma_4(G))$. It would be desirable to get a more refined information than the degree of commutativity; in particular, we would like to know the values of $f(a) := \min\{i : \mathrm{nilp}(\gamma_i(G)) \geq a\}$ for $a = 1, ..., n-1$.

5.2 Main result for ECF-groups and p-groups of maximal class

We state what happens when we drop the assumption of CF-groups having exponent p? It turns out that we can still say something useful about the exponent of the subgroups appearing in the lower central series.

Definition. A CF-group G is said to be an ECF-group if the quotient $G/\gamma_2(G)$ is elementary abelian.

Even though G may not have exponent p, the subgroups appearing in the lower central series have exponent p.

Theorem 5.3. [Bla58, Theorem 5.3] Suppose that G is a p-group that is an ECF-group of nilpotency class n and coclass c. If $4 \leq n \leq p+1$, then, $G/\gamma_n(G)$ and $\gamma_2(G)$ are of exponent p.

Corollary 5.1. Let G be as in Theorem 5.3. Then, G has exponent at most p^2.

Proof. Let $g \in G$. Then, $g^p \in \gamma_n(G)$. Since $\gamma_n(G) \cong C_p$, g^p is of exponent p and thus, g is of exponent p^2. □

Theorem 5.4. Let G be a p-group that is an ECF-group of nilpotency class $n < p$, coclass c and degree of commutativity d.

Proof. Let $H_a := \gamma_{f(a)}(G)$ where $f(a) = \max\{2, \frac{(n+1)-d}{a+1}\}$. In particular, $H_a = \gamma_2(G)$ for all $a \geq \frac{n-1}{2+d}$.

$$X_N := \left\{ \sum_{i \in \mathbb{Z}_0(p) \cap [0,N]} D_i t^{-i} : D_N \neq 0, D_i \in \begin{cases} L(H_a) \otimes \mathbb{F}_q & N/(a+1) < i \leq N/a \text{ for all } a = 1, ..., \lfloor \frac{n-1}{2+d} \rfloor - 1, \\ L(\gamma_2(G)) \otimes \mathbb{F}_q & 0 \leq i \leq \frac{N}{\lfloor (n-1)/(2+d) \rfloor} \end{cases} \right\}$$

A similar computation can be used to show that

$$|X_N| \gg q^{N(An - \frac{1}{\lfloor (n-1)/(d+2) \rfloor})(1-1/p)}.$$

□

6 Main results for the Heisenberg group

In this section, we analyze the distribution of the last jumps in the Heisenberg group $\mathcal{H}_3(\mathbb{F}_p)$. A somewhat surprising consequence of the discussion below is that the nonintegral ramification breaks seem to be the main terms contributing to these counts. We also get lower bounds which get very close to the upper bounds given by [GS25].

Let $U_{L/K}$ denote the set of jumps in the ramification subgroups of $\mathrm{Gal}(L/K)$.

Theorem 6.1. Let $a, b \not\equiv 0 \pmod{p}$, $a+b \not\equiv 0 \pmod{p}$ and $a < b$.

$$\#\{L : \mathrm{Gal}(L/K) \cong \mathcal{H}_3(\mathbb{F}_p), U_{L/K} = \{a, b, b+a/p\}\} \gg q^{(1-1/p)(2b+a)}.$$

Remark. In this particular setting, this lower bound gets pretty close to the upper bound in [GS25, Proposition 6.2]. Keeping track of the dependency on the residue classes of a, b gives us $\#\{L : \mathrm{Gal}(L/K) \cong \mathcal{H}_3(\mathbb{F}_p), U_{L/K} = \{a, b, b+a/p\}\} \geq q^{2b+a-2\lfloor b/p \rfloor - \lfloor a/p \rfloor}$.

If $a, a+1 \not\equiv 0 \pmod{p}$ then we have $\#\{L : \mathrm{Gal}(L/K) \cong \mathcal{H}_3(\mathbb{F}_p), \mathrm{lastjump}(L/K) = a(1+1/p)\} \geq Cq^{3(a-\lfloor a/p \rfloor)+2}$ (where $C = (1-1/q)^2$).

From [GS25], we have this count is $\leq q^{3(\lceil a(1+1/p) \rceil - \lceil \frac{a(1+1/p)}{p} \rceil)}$. In other words, the exponent is $3(a + \lceil a/p \rceil - \lceil a/p + a/p^2 \rceil)$

Remark. [EK23, Proposition 5.2] showed the existence of such $\mathcal{H}_3(\mathbb{F}_p)$-extensions with these ramification breaks.

Proof. We describe an element $x = \sum_i D_i t^{-i}$ with constraints on D_i that give us these ramification breaks. Take two elements $y_1, y_2 \in L(\mathcal{H}_3(\mathbb{F}_p))$ with $[y_1, y_2]_{\mathbf{LieAlg}} = z \neq 0$.

Let $L_1 = L(\mathcal{H}_3(\mathbb{F}_p)) = y_1\mathbb{F}_p \oplus y_2\mathbb{F}_p \oplus z\mathbb{F}_p, L_2 = y_2\mathbb{F}_p \oplus z\mathbb{F}_p, L_3 = L(Z(\mathcal{H}_3(\mathbb{F}_p))) = z\mathbb{F}_p$

Let $D_i \in \begin{cases} L_1 \otimes \mathbb{F}_q & i \in [0, a] \cap \mathbb{N}^0(p), \\ L_2 \otimes \mathbb{F}_q & i \in [a+1, b] \cap \mathbb{N}^0(p). \end{cases}$

We further require that $D_a \notin L_2, D_b \notin L_3$ which gives us that $[D_a, D_b] \neq 0$

We now need to assign $D_i \in L_3 \otimes \mathbb{F}_q$ appropriately for $i \in [a+b, b+1] \cap \mathbb{N}^0(p)$. In particular, for every $i \in [a+b, b+1] \cap \mathbb{N}^0(p)$,

$$D_i := \frac{-1}{2i} \sum_{\substack{i=m+n, \\ m,n \in \mathbb{N}^0(p)}} m[D_m, D_n].$$

This guarantees that $\mathcal{F}_{\gamma,-N} = 0$ for all $\gamma \in [a+b, b+1] \cap \mathbb{N}^0(p)$. On the other hand, $\mathcal{F}_{b+a/p,-N} = \frac{1}{2}b[D_b, \sigma^{-1}D_a]$ and $[D_b, \sigma^{-1}D_a]$ is a nonzero $\mathbb{F}_q$-scalar multiple of $0 \neq [y_1, y_2] \in Z(\mathcal{H}_3(\mathbb{F}_p))$.

We now estimate the number of choices we made. There are $a - \lfloor a/p \rfloor$ indices in $[0, a] \cap \mathbb{N}^0(p)$ and $(b - \lfloor b/p \rfloor) - (a - \lfloor a/p \rfloor)$ indices in $[a+1, b] \cap \mathbb{N}^0(p)$, For D_a, D_b we made $q^3(1-1/q)$ and $q^2(1-1/q)$ choices respectively and for all other indices $i \in [0, a] \cap \mathbb{N}^0(p)$ and $i \in [a+1, b] \cap \mathbb{N}^0(p)$ we made q^3, q^2 choices respectively. In particular, we have

$$(1-1/q)^2(q^3)^{a-\lfloor a/p \rfloor} \cdot (q^2)^{(b-\lfloor b/p \rfloor)-(a-\lfloor a/p \rfloor)} = (1-1/q)^2 \cdot q^{2b+a-2\lfloor b/p \rfloor - \lfloor a/p \rfloor}.$$

□

7 Application to p-ranks of G-covers

7.1 Low Ramification Breaks

We will assume $\mathrm{Gal}(F/K)$ is a p-group of nilpotency class $< p$ to be in the setting of Abrashkin's nilpotent Artin-Schreier theory. Recall the following things about ramification groups of Galois extension of local fields F/K.

1. F/K is unramified if F/K has $G^1(F/K) = \{1\}$.
2. F/K is (wildly) ramified if $G^0(F/K) = G^1(F/K) \neq \{1\}$. This follows as the quotient $G^0(F/K)/G^1(F/K)$ is coprime to p.

Suppose L/F is wildly ramified. One thing we need to address is how low these ramifications can be. The reason why we are interested in this is because of the problem of finding lower bounds on the p-coranks requires us to have small breaks in the ramification groups.

Lemma 7.1. Let G be a p-group of nilpotency class $< p$. Let $x, y \in G$ be two elements of order p. Then, $x \circ y$ is an element of order p.

Proof. Let L be the corresponding Lie algebra of G. The p-th power of the group element $x \circ y$ is $p(x \circ y) = p(x + y + \frac{1}{2}[x, y] + ...) = 0$. □

Proposition 3. Let $k = \overline{\mathbb{F}_p}$ and let G be a finite p-group with nilpotency class $< p$.

There exists an ordinary connected G-cover of $\mathbb{P}^1_k$ if and only if the exponent of G is p.

Proof. ($\Longleftarrow$)

Let L be the corresponding finite $\mathbb{Z}_p$-Lie algebra of the group G.

For a cover to be ordinary, we require that the last jump of the ramification groups with lower numbering are either 1 or 0 at every place of $k(t)$.

If this étale algebra extension $F/k(t)$ defines an extension of fields, by Lemma 7.1, we have that there is at least one place $\mathfrak{p}$ of $k(t)$ in which the decomposition group $\mathrm{Gal}(F_{\mathcal{P}}/k(t)_{\mathfrak{p}})$ is isomorphic to a p-group of exponent $> p$. That is to say that the elements of order p cannot generate a subgroup of exponent higher than p.

We now claim that the last break in the ramification groups with lower numbering cannot be 0 or 1 at the place $\mathfrak{p}$. If it were 0, the local extension would be unramified and thus, the decomposition group at place $\mathfrak{p}$ should be trivial.

Suppose $\sum_{b \leq B} D_b t^{-b} \in L \otimes D^0$ be an element parametrizing the local extension $F_{\mathfrak{P}}/K_{\mathfrak{p}}$. Suppose $\mathrm{Gal}(F_{\mathfrak{P}}/K_{\mathfrak{p}})$ is of cardinality p^r where $r \geq 2$.

Consider, $B := \inf\{b > 0 : o(D_b) = p^r\}$; the set is certainly non-empty.

The claim is that $\mathcal{F}_{\gamma,-N} \neq 0$ whenever $\gamma = B$. This happens because the Lie bracket terms appearing in $\mathcal{F}_{B,-N}$ are all of order strictly less than p^r.

Thus, $p^{r-1}\mathcal{F}_{B,-N} = p^{r-1}D_B \neq 0$

($\Longrightarrow$) Suppose G is a group of exponent p. Let $H_1, ..., H_n$ be subgroups of order p which generate G. For each i, suppose l_i is an elements in $G(L)$ which generate the subgroup H_i.

Take n distinct places of $\mathbb{F}_q(t)$ for q sufficiently large. Call these places $\mathfrak{p}_i = (p_i)$.

Then, $\sum_{i=1}^{n}(l_i \otimes \frac{1}{p_i})$ is a representative of a class in $G(L \otimes W(K^{\text{perf}}))//_{G(L\otimes W(K^{\text{perf}}))}$ which generates an ordinary G-extension. □

Notations

L	$\mathbb{Z}_p$-Lie algebra of nilpotency class $< p$	Theorem 3.1
$\mathcal{G}(L)$	the finite p-group associated algebra L	Theorem 3.1
$[b_1, b_2, ..., b_n]$	Repeated Lie brackets $[\dots[b_1, [b_2, ..., b_n]\dots]$	
$Z(L)$	the center of the Lie algebra L	
$Z(G)$	the center of a group G	
$\exp(G)$, $\text{nilp}(G)$	exponent of a group G, nilpotency class of a group G	
$O(R), W(R)$	the Cohen ring of R and Witt ring of R	
$\mathbb{Z}^0(p), \mathbb{Z}(p)$	$\mathbb{Z} \setminus p\mathbb{Z} \cup \{0\}$, $\mathbb{Z} \setminus p\mathbb{Z}$	[Abr98, pg. 857]
$\mathbb{N}^0(p), \mathbb{N}(p)$	$\mathbb{N} \setminus p\mathbb{N} \cup \{0\}, \mathbb{N} \setminus p\mathbb{N}$	
$G^i(\mathfrak{P}/\mathfrak{p}), G_i(\mathfrak{P}/\mathfrak{p})$	ramification groups with upper/lower numbering	
$\mathcal{D}^0$	$\mathcal{D}^0 = \sum_{a\in\mathbb{N}^0(p)} W(\mathbb{F}_q)t^{-a}$	
$\mathcal{H}_3(\mathbb{F}_p)$	the Heisenberg group over $\mathbb{F}_p$, $\left\{ \begin{bmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{bmatrix} : a, b, c \in \mathbb{F}_p \right\}$	
$\mathbb{F}_p^n \rtimes \mathbb{F}_p$	a subgroup of affine transformations of $\mathbb{F}_p^n$	§4 Example 1
$\gamma_i(G)$	lower central series with the convention that $\gamma_0(G) = G, \gamma_2(G) = [G, G]$	

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