

Wildly ramified covers and Artin-Schreier conductor

Eiki Norizuki

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Abstract

Let $\mathbb{F}_q$ be a finite field of characteristic p . Unlike the counts by genus investigated by [San20], we show that a positive proportion of Artin-Schreier covers are ordinary when we order them by the Artin-Schreier conductor which was studied by Gundlach [Gun26]. We also study the p -rank of G -covers where G is an abelian p -group and show bounds on the attainable p -rank.

1 Introduction

When we count Artin-Schreier curves by their genus, the proportion of those curves that are ordinary is zero for all odd characteristics by a result of Sankar. This arithmetical statistical fact is related to the works of Pries and Zhu [ZP12] where they showed that the moduli space of Artin-Schreier curves of genus g is not irreducible for $p > 2$.

Gundlach [Gun26] showed that an invariant called the Artin-Schreier conductor of an extension of function fields exhibits nice properties for wild extensions. To be more precise, he showed that the generating function where the coefficients are the number of G -extensions of $K = \mathbb{F}_q(t)$ is meromorphic to the entire complex plane and furthermore rational; this is a property not usually exhibited in similar counts with discriminants or the classical conductor. In this article, we will use the framework in Gundlach's paper to show that a positive proportion of $\mathbb{Z}/p\mathbb{Z}$ -covers of $\mathbb{P}^1$ defined over $\mathbb{F}_q$ are ordinary when we count by the Artin-Schreier conductor. We will also state analogous results on the p -ranks for other cyclic $\mathbb{Z}/p^e\mathbb{Z}$ -cover and elementary abelian p -groups $G = (\mathbb{Z}/p\mathbb{Z})^m$. The generating functions which we consider turns out to be rational, further hinting to Gundlach [Gun26]'s question that there may be a moduli space interpretation of a G -cover with a fixed Artin-Schreier conductor.

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2 Setup

2.1 Class Field Theory

We will recall some definitions and results from class field theory. Let $K = \mathbb{F}_q(t)$. Let M_K denote the places of K .

Theorem 2.1 (Local Artin Reciprocity). Let K be a global field and $K_{\mathfrak{p}}$ be the local field obtained by completing at $\mathfrak{p}$. The local Artin reciprocity is the map,

$$\theta_{\mathfrak{p}} : K_{\mathfrak{p}}^{\times} \rightarrow \text{Gal}(K_{\mathfrak{p}}^{ab}/K_{\mathfrak{p}}).$$

A non-trivial fact from class field theory is that we can define a global reciprocity map by taking the local reciprocity map at each place of K .

Theorem 2.2 (Global Artin Reciprocity). Let $\mathbb{A}_K^{\times}$ be the group of ideles on a global field K . The global reciprocity map

$$\theta_K : \widehat{\mathbb{A}_K^{\times}/K^{\times}} \rightarrow \text{Gal}(K^{ab}/K).$$

is an isomorphism.

We have a concrete description of the adèle class group as $(\prod_{\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{p}}}^{\times})/\mathbb{F}_q^{\times} \times \widehat{\mathbb{Z}} \cong \widehat{\mathbb{A}_K^{\times}/K^{\times}}$. For a finite abelian p -group G , $\text{Hom}(\text{Gal}(K^{ab}/K), G)$ parametrizes all G -extensions of K . Combining the above discussion, a G -extension of K corresponds to a map $\varphi : (\prod_{\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{p}}}^{\times})/\mathbb{F}_q^{\times} \times \widehat{\mathbb{Z}} \rightarrow G$. Since, there are $|G|$ -maps from $\widehat{\mathbb{Z}} \rightarrow G$, our goal is to count maps $\varphi : (\prod_{\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{p}}}^{\times})/\mathbb{F}_q^{\times} \rightarrow G$. We have local maps $\varphi_{\mathfrak{p}} : \mathcal{O}_{K_{\mathfrak{p}}}^{\times} \rightarrow G$ and to understand the ramification filtration at the place $\mathfrak{p}$, we need the following result on the higher unit groups $U_{\mathfrak{p}}^k := 1 + \mathfrak{m}_{\mathfrak{p}}^k$. See [Gun26, Lemma 4.1] for a proof of the theorem.

Theorem 2.3. Let $\mathfrak{p}$ be a place of $\mathbb{F}_q(t)$ and suppose that $q^{\deg \mathfrak{p}} = p^d$. There exists an isomorphism $\alpha_{\mathfrak{p}} : \prod_{p \nmid i, i \geq 1} \mathbb{Z}_p^d \rightarrow U_{\mathfrak{p}}^1$ such that the preimage of $U_{\mathfrak{p}}^k \subset U_{\mathfrak{p}}^1$ is isomorphic to the subgroup $\prod_{p \nmid i, i \geq 1} p^{\nu(k/i)} \mathbb{Z}_p^d$.

2.2 Curves and Function Fields

We will frequently switch between thinking about curves and their function fields throughout the rest of this paper. A curve over a base field k will be an irreducible one-dimensional variety; in particular it is connected. Let C be a smooth projective curve over some base field k . Denote the function field of C by $k(C) = \text{Frac} k[C]$. For us, we will mostly be thinking about the case when $C = \mathbb{P}^1$ and when $k = \mathbb{F}_q$. Recall that there is a canonical bijection between the following two sets;

$$\{\pi : C' \rightarrow C, \text{finite nonconstant map of (smooth projective) curves}\} \\ \leftrightarrow \{\text{finite field extensions of fields of transcendence degree one, } k(C')/k(C)\}.$$

Let $k[C]$ be the ring of integers of $k(C)$. In particular, $\text{Spec } k[C]$ is a affine normal curve with C as its projective closure. Primes of $k[C] = \mathbb{F}_q[t]$ correspond to closed points on the affine line $\mathbb{A}^1 = \text{Spec } \mathbb{F}_q[t]$ which is a subscheme of $\mathbb{P}^1$. If we are interested in the closed points on $\mathbb{P}_k^1$, we need to study the *places* of $\mathbb{F}_q[t]$. The point at infinity of $\mathbb{P}^1$ corresponds to the prime $\mathfrak{p} = (1/t)$ of $\mathbb{F}_q[1/t]$ which agrees with the place represented by $f \mapsto -\deg(f)$ on $\mathbb{F}_q(t)$.

Similarly, if we look at finite etale extensions of $k(C)$ (direct sum of finite field extensions of $k(C)$), we obtain the following canonical bijection,

$$\{\pi : \coprod_{i=1}^n C_i \rightarrow C, \text{finite nonconstant map}\} \leftrightarrow \{\text{finite etale extensions } \bigoplus_{i=1}^n k(C_i)/k(C)\}.$$

Let G be a finite abelian group. The G -extensions of K are in bijection with $\text{Hom}(\text{Gal}(K^{ab}/K), G)$. When $\varphi \in \text{Hom}(\text{Gal}(K^{ab}/K), G)$ is surjective, this is the usual Galois correspondence for fields; namely, take the fixed field $L = (K^{\text{sep}})^{\ker \varphi}$. When $\varphi \in \text{Hom}(\text{Gal}(K^{ab}/K), G)$ is not surjective, the fixed field $L = (K^{\text{sep}})^{\ker \varphi}$ still defines a field extension of K with Galois group $H = \text{im}(\varphi) \subset G$. The etale algebra, $A = \prod_{G/H} L$ is a G -extension of K . It is worth noting that $\text{Aut}_K(A)$ need not equal G ; in fact $G \subset \text{Aut}_K(A)$ with equality if A is an etale algebra.

Example. If $\varphi \in \text{Hom}(\Gamma_K, \mathbb{Z}/3\mathbb{Z})$ is the zero map, it defines the $\mathbb{Z}/3\mathbb{Z}$ -extension $A = \prod_{g \in \mathbb{Z}/3\mathbb{Z}} K_g$. However, $\text{Aut}_K(A) = S_3$.

Definition (G -cover). Let C be a curve over k and X is a one-dimensional projective (possibly disconnected) variety. X is said to be a G -cover of C if $\pi : X \rightarrow C$ induces an etale extension $k(X)/k(C)$ with Galois group isomorphic to G .

2.3 Arithmetic Invariants of Curves

The main arithmetic invariant that we are interested in is the p -rank. Given an abelian variety A over a field of positive characteristic, the p -rank records information about the p -torsion of A which we denote $A[p]$. More precisely, the p -rank denoted $f = f_A$ is given

by $f_A = \dim_{\mathbb{F}_p} \text{Hom}(\mu_p, A[p])$. For a smooth projective curve C over a field of positive characteristic, the p -rank of C (denoted f_C) is the p -rank of $J(C)[p]$.

More often for our purposes, we will be interested in the p -corank which we define to be the difference between the genus and the p -rank (i.e. $g_C - f_C$). The reason for using this is that the ordinary curves have a p -corank of 0.

2.4 Ramification groups in lower and upper numbering

We will need both the lower and upper numbering for ramification groups. The lower numbering will be needed for the computation of the p -corank and the upper numbering is needed for the computation of the Artin-Schreier conductor.

Let L/F be a finite Galois extension of local fields with $\pi = \pi_L$ the uniformizer of L . The ramification groups with lower numbering are defined as $G_i = \{g \in G : v_\pi(gx - x) \geq i + 1\}$. The Herbrand function is a piecewise linear function given by the integral $\psi(x) = \int_0^x \frac{1}{[G_0 : G_t]} dt$. The ramification groups with upper numbering is simply a relabeling of the ramification groups with lower numbering by this Herbrand function. More precisely, $G_i = G^{\psi(i)}$. The Hasse-Arf Theorem states that the breaks in the upper ramification groups occur at integer values for abelian extensions L/F .

Definition. Let L/F be an abelian extension of global function fields. The Artin-Schreier conductor of L/F is defined as

$$\text{asc}(L/K) = \sum_{\mathfrak{p} \in M_K} \text{lstjmp}_{\mathfrak{p}}(L/K) \cdot \mathfrak{p}$$

where $\text{lstjmp}_{\mathfrak{p}}(L/K)$ is the last jump in the upper numbering ramification filtration at the place $\mathfrak{p}$ of F .

Remark. It is worth noting down that the degree of the Artin-Schreier conductor is independent of base change. More precisely, if L/K is a totally ramified extension of global function fields defined over $\mathbb{F}_q$, then $\deg(\text{asc}(L/K)) = \deg(\text{asc}(L \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}) / (K \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}))$.

3 Main Results

Let $\mathbb{F}_q$ be a field of characteristic p . Denote the rational function field $\mathbb{F}_q(t)$ by K . Unless stated otherwise, all curves are smooth and projective over $\mathbb{F}_q$ and all covers $C \rightarrow \mathbb{P}^1$ that we study are connected and etale. For a place $\mathfrak{p}$ of K , denote the degree of the place by $\deg \mathfrak{p}$.

3.1 The arithmetic statistics of p -corank of $\mathbb{Z}/p^e\mathbb{Z}$ -covers of $\mathbb{P}^1$

Lemma 3.1. The generating function for the number of isomorphism classes of etale $\mathbb{Z}/p^e\mathbb{Z}$ -covers of $\mathbb{P}^1$ with Artin-Schreier conductor k is

$$F_K^{\text{asc}}(X) = \frac{(1-X)(1-q^{p^e+1}X^{p^e})}{(1-q^2X)(1-q^{p^e-1}X^{p^e})}.$$

In particular, the number of isomorphism classes of $\mathbb{Z}/p^e\mathbb{Z}$ -covers of $\mathbb{P}^1$ with Artin-Schreier conductor $\leq k$ is

$$\#\{L : \text{Gal}(L/K) = \mathbb{Z}/p^e\mathbb{Z}, \deg(\text{asc}(L/K)) \leq k\} \sim \frac{1 - q^{-p^e+1}}{1 - q^{-p^e-1}} \cdot q^{2k}.$$

Proof. This is stated in [Gun26, 4.3] but we will make the constants explicit. □

Lemma 3.2. Let $C \rightarrow \mathbb{P}^1$ be a G -cover defined over $\mathbb{F}_q$, where G is a p -group.

$$g_C - f_C = \frac{1}{2} \sum_{\mathfrak{p}} \left(\sum_{i \geq 2} (\#G_i(\mathcal{P}/\mathfrak{p}) - 1) \right) \cdot \deg \mathfrak{p}.$$

Proof. Let L/K be a G -cover and S be the set of places that ramify. Recall that the genus of L is given by Riemann-Hurwitz formula $2g - 2 = |G|(-2) + \deg R$ where R is the ramification divisor. For wild covers, the ramification divisor can be computed as the sum of local differentials (see [Sti93, Theorem 3.8.7]),

$$\deg R = \sum_{\mathfrak{p}} \left(\sum_{i \geq 0} (\#G_i(\mathcal{P}/\mathfrak{p}) - 1) \right) \cdot \deg \mathfrak{p}.$$

For all the places that are ramified, they are wildly ramified. In particular, the ramification group with lower numbering $G_1(\mathcal{P}/\mathfrak{p})$ is not trivial. Also, note that G_0/G_1 has cardinality coprime to p . This means that $G_1 = G_0$. On the other hand, the p -rank of L can be computed by the Deuring-Shafarevich formula (see [Cre84, Corollary 1.8]),

$$f - 1 = |G|(-1) + \sum_{\mathfrak{p}} (e(\mathcal{P}/\mathfrak{p}) - 1) \cdot \deg \mathfrak{p}.$$

Since, the ramification index $e(\mathcal{P}/\mathfrak{p})$ is equal to the size of the decomposition group $G_0(\mathcal{P}/\mathfrak{p})$, we obtain that the genus and p -rank are equal if and only if $\sum_{i \geq 2} (\#G_i(\mathcal{P}/\mathfrak{p}) - 1) = 0$ for every place $\mathfrak{p}$ or in other words, $G_i(\mathcal{P}/\mathfrak{p}) = 1, i \geq 2$ for every place $\mathfrak{p}$. □

Corollary 3.3. Let C be a curve over $\mathbb{F}_q$ and suppose that C is a G -cover of $\mathbb{P}^1$ with G being an abelian p -group. C is ordinary if and only if its ramification group $G_i(\mathcal{P}/\mathfrak{p})$ is trivial for $i \geq 2$ at all places $\mathfrak{p}$.

Remark. Let G be a p -group of order p^n and suppose that $l_1 \leq \dots \leq l_n$ are breaks in the lower numbering ramification groups (counted with multiplicity). Suppose that $l_i \geq 2$ for some i and let $k = \min\{i : l_i \geq 2\}$. An equivalent formula for 3.2 is

$$g - f = (l_k - 2) \cdot p^{k-1} + \sum_{i=k+1}^{n-1} (l_{i+1} - l_i) \cdot p^{n-i}.$$

Theorem 3.4. Define the generating function counting the number of isomorphism classes of etale $\mathbb{Z}/p^e\mathbb{Z}$ -covers of $\mathbb{P}^1$ that are ordinary by the Artin-Schreier conductor,

$$F_{K,\text{ord}}(X) := \frac{1}{|G|} \sum_{k \geq 0} \#\{C : \text{Gal}(k(C)/K) \cong \mathbb{Z}/p^e\mathbb{Z}, C \text{ is ordinary, } \text{asc}(k(C)/K) = k\} \cdot X^k.$$

This generating function is of the form,

$$F_{K,\text{ord}}(X) = \prod_{\mathfrak{p}} (1 + q^{\deg \mathfrak{p}} X^{\deg P}) = \frac{(1 + qX)(1 - q^3 X^2)}{1 - q^2 X}$$

Remark. It is worth pointing out that this generating function has no dependency on the order of the cyclic group $\mathbb{Z}/p^e\mathbb{Z}$ which may be surprising at first.

Proof. By our characterization in 3.3, we know that $G^i(\mathfrak{P}/\mathfrak{p}) = 1$ for every $i \geq 2$ and for every place $\mathfrak{p}$. The function $F_{K,\text{ord}}(X)$ admits an Euler product, $F_{K,\text{ord}}(X) = \prod_{\mathfrak{p}} F_{K_{\mathfrak{p}},\text{ord}}(X)$ where the local generating functions are $F_{K_{\mathfrak{p}},\text{ord}}(X) = \sum_{\varphi: U_{\mathfrak{p}}^1/U_{\mathfrak{p}}^2 \rightarrow G} X^{\text{asc}(\varphi)}$. The quotient $U_{\mathfrak{p}}^1/U_{\mathfrak{p}}^2 \cong (\mathbb{Z}/p\mathbb{Z})^d$ where $q^{\deg \mathfrak{p}} = p^d$. Fix any $\mathbb{Z}/p\mathbb{Z}$ -basis $e_1, \dots, e_d$ of $(\mathbb{Z}/p\mathbb{Z})^d$. Since each of the e_i should map to an element of order p or 1, each element has precisely p elements in $G = \mathbb{Z}/p^n\mathbb{Z}$ that it can map to. Thus, there are exactly p^d maps and have local Artin-Schreier conductor = 1. Hence, $F_{K_{\mathfrak{p}},\text{ord}}(X) = 1 + q^{\deg \mathfrak{p}} X^{\deg \mathfrak{p}}$. □

Since, the above generating function is rational, we can now obtain an exact formula for the number of $\mathbb{Z}/p^n\mathbb{Z}$ -covers of $\mathbb{P}^1$ with a given Artin-Schreier conductor. Noting that this count is for etale $\mathbb{Z}/p^n\mathbb{Z}$ -covers which may be disconnected. If we restrict to the case where the cover is an actual curve, we get the following.

Corollary 3.5. There are $q^{2k} - q^{2k-2}$ ordinary Artin-Schreier curve (i.e. connected $\mathbb{Z}/p\mathbb{Z}$ -covers of $\mathbb{P}^1$) of Artin-Schreier conductor k . There are no ordinary $\mathbb{Z}/p^e\mathbb{Z}$ -covers of $\mathbb{P}^1$ for $e > 1$.

In particular, this means that when we count by the Artin-Schreier conductor, a positive proportion of Artin-Schreier curves are ordinary.

Corollary 3.6. The proportion of ordinarity is positive when counted by the Artin-Schreier conductor. Namely,

$$\lim_{k \rightarrow \infty} \frac{\#\{C \rightarrow \mathbb{P}^1, \text{asc}(C) = k, \text{Gal}(k(C)/\mathbb{F}_q(t)) = \mathbb{Z}/p\mathbb{Z}, C \text{ is ordinary}\}}{\#\{C \rightarrow \mathbb{P}^1, \text{asc}(C) = k, \text{Gal}(k(C)/\mathbb{F}_q(t)) = \mathbb{Z}/p\mathbb{Z}\}} = \frac{1 - q^{-p-1}}{1 - q^{-p+1}}$$

3.2 Relationship to the genus

In this section, we attempt to give a geometric explanation for what is going on when $G = \mathbb{Z}/p\mathbb{Z}$ relating to Pries and Zhu's results [ZP12] on the geometry of $\mathcal{AS}_{g,f}$, the moduli space of Artin-Schreier curves of genus g , p -rank f . This situation is simpler in many ways because the filtration of the ramification group is only determined by the last jump and the lower and upper ramification groups are the same. Thus, the relationship between the genus, p -rank and Artin-Schreier conductor is simpler than more general wild covers. For an Artin-Schreier cover L/K defined over $\mathbb{F}_q$, we have that

$$g(L) = \frac{p-1}{2} \left(-2 + \sum_{P \text{ ramified}} (\text{lstjmp}_P(L) + 1) \cdot \deg P \right).$$

The ramification divisor of the extension L/K which we denote $B(L) := \sum_{P, \text{ramified}} P$ and the Artin-Schreier conductor also determine the genus g .

$$g(L) = \frac{p-1}{2} (-2 + \deg \text{asc}(L) + \deg B(L))$$

Suppose we are interested in studying Artin-Schreier covers with Artin-Schreier conductor $= k$. Since, $\deg B(L) \leq \deg \text{asc}(L)$, we conclude that we are interested in Artin-Schreier covers with genus $g = \frac{p-1}{2}(-2 + k + b)$ and p -rank $f = \frac{p-1}{2}(-2 + 2b)$ where $b \leq k$. Thus, an Artin-Schreier covers over $\mathbb{F}_q$ with Artin-Schreier conductor $= k$ is an element of

$$X(\mathbb{F}_q) := \coprod_{b \leq k} \mathcal{AScov}_{\frac{p-1}{2}(-2+k+b), \frac{p-1}{2}(-2+2b)}(\mathbb{F}_q)$$

Note that $\dim X = 2k$ and it has a unique irreducible component of that dimension. See [ZP12, Theorem 1.1 and Lemma 3.1] on the dimension of these moduli spaces.

3.3 Lower bound on the p -corank of a $\mathbb{Z}/p^e\mathbb{Z}$ -cover of $\mathbb{P}^1$

Proposition 3.7. Let $G = \mathbb{Z}/p^e\mathbb{Z}$ where $e > 1$.

If C is a curve that is a (connected) $\mathbb{Z}/p^e\mathbb{Z}$ -cover of $\mathbb{P}^1$, then the p -corank is bounded below by

$$g - f \geq \frac{p(p^{e-1} - 1)(p^e - 1)}{p + 1}$$

The number of such curves C over $\mathbb{F}_q$ with Artin-Schreier conductor $= k$ is asymptotically,

$$\sim Aq^{-p^{e-1}+1}q^{2k}$$

where $A = (1 + 1/q)(1 - 1/q^2)$.

Proof. To attain the least p -corank, we need all prime $P' \neq P$ to have $\text{lstjmp}_{P'}(\varphi) = 1$ and $\text{lstjmp}_P(\varphi) = k$ (where k is to be determined). Without loss of generality, we may assume P is a degree one prime.

If $\varphi : \widehat{\mathbb{A}_K^\times/K} \rightarrow G$ defines a field extension of K , we need φ to be a surjection. In particular, the local map $\varphi_P : U_P^1/U_P^{k+1} \rightarrow G$ should be a surjection for some place P . By Theorem 3.7, there exists an isomorphism α

$$U_P^1/U_P^{k+1} \cong \prod_{p \nmid i, i \geq 1} (\mathbb{Z}/p^{\nu((k+1)/i)}\mathbb{Z})$$

so we now seek the least k such that $\nu(k+1) = e$. The least such k is $k = p^{e-1}$. For $\varphi_P \circ \alpha : \prod_{p \nmid i, i \geq 1} \mathbb{Z}_p^d \rightarrow G$ to be a surjection, we require the first term in the product to map surjectively to G .

Let $\text{jmp}_{P,i}(\varphi) := \min\{k : \varphi(U^k) \subset G[p^{e-i}]\}$. In particular, $\text{jmp}_{P,e}(\varphi) = \text{lstjmp}_P(\varphi)$. Let $b_1 \leq b_2 \leq \dots \leq b_e$ be the breaks in the upper filtration groups. The breaks in the ramification groups with lower numbering are given by $c_{1,P} \leq c_{2,P} \leq \dots \leq c_{e,P}$ (in fact these inequalities are strict) where $c_{i,P} = \sum_{l=1}^i b_l p^{l-1}$. The p -corank is therefore given by

$$\sum_P \left((c_{1,P} - 1)(p^e - 1) + \sum_{l=2}^e (c_{l,P} - c_{l-1,P})(p^{e-l+1} - 1) \right)$$

Let us now determine where the breaks in the upper filtration groups are with respect to this map φ .

$$\varphi(U^l) = \varphi \circ \alpha \left(\prod_{i \geq 1, p \nmid i} p^{\nu(l/i)} \mathbb{Z}_p \right)$$

But since the map acts trivially on all the terms in the product except for the first term, we get that $\varphi(U_P^l) = p^{\nu(l)}G$. In particular, the breaks in the upper filtration correspond to the breaks in the value $\nu(l)$ for $l = 1, p, p^2, \dots, p^{e-1}$. The breaks in the upper numbering of the ramification groups occur at $1 \leq p \leq p^2 \leq \dots \leq p^{e-1}$. Converting this to lower numbering, the breaks occur at $1 \leq 1 + (p-1) \cdot p \leq \dots \leq 1 + \sum_{l=1}^e (p^l - p^{l-1}) \cdot p^l$. In other words, we have $p^l - p^{l-1}$ many integral indices where the upper ramification groups $G^i(\mathcal{P}/\mathfrak{p})$ have cardinality p^{e-l} for $l = 1, \dots, e-1$.

Using Lemma 3.2, we will get,

$$\begin{aligned} \sum_{l=1}^{e-1} p^l (p^l - p^{l-1}) (p^{e-l} - 1) &= (p-1) \sum_{l=1}^{e-1} (p^{e+l-1} - p^{2l-1}) = (p-1)p^{e-1} \cdot \frac{p^e - p}{p-1} - (p-1) \frac{p^{2e-1} - p}{p^2 - 1} \\ &= p^e (p^{e-1} - 1) - \frac{p(p^{e-1} - 1)(p^{e-1} + 1)}{p+1} = (p^{e-1} - 1) \left(p^e - \frac{p(p^{e-1} + 1)}{p+1} \right) = (p^{e-1} - 1)p \left(\frac{p^e - 1}{p+1} \right) \end{aligned}$$

Now we will compute the number of such $\mathbb{Z}/p^e\mathbb{Z}$ -extensions of K with a prescribed Artin-Schreier conductor.

We count the number of maps $\varphi : \mathbb{A}_K/K^\times \rightarrow G$ with the prescribed local conditions and the number of those maps that land in $G[p^{e-1}]$. The difference is what we want to compute.

Firstly, we look at the generating function for the etale G -extensions of K with the prescribed local conditions. To be more precise, the local conditions are that there is a degree one prime P such that all ramified prime $P' \neq P$ has $\text{lstj}_{P'}(\varphi) = 1$ and at P , it has jumps at $1, p, p^2, \dots, p^{e-1}$. For simplicity, we will call this local condition T . Let us call this generating function $F_{K,G,\text{smallest p-corank}}^{\text{asc}}$.

$$F_{K,G,\text{smallest p-corank}}^{\text{asc}} = q^{p^{e-1}} X^{p^{e-1}+1} \cdot \frac{1 - q^3 X^2}{1 - q^2 X}$$

Essentially what we are doing is taking the generating function for the ordinary (not necessarily connected) G -covers and modifying the local factor at a degree one prime with the local factor that counts the above local conditions. Therefore, the number of G -covers with the local conditions T and Artin-Schreier conductor $= k$ is asymptotically,

$$\sim q^{p^{e-1}} (q^2 - q) q^{2k - p^{e-1} - 1} = A q^{-p^{e-1}+1} q^{2k}$$

where $A = (1 - 1/q)$.

□

Remark. By Lemma 3.1, the number of all $\mathbb{Z}/p^e\mathbb{Z}$ -covers (including disconnected ones) with Artin-Schreier conductor $= k$ is

$$\sim \frac{1 - q^{-p^e+1}}{1 - q^{-p^e-1}} q^{2k}$$

Therefore, the number of all connected $\mathbb{Z}/p^e\mathbb{Z}$ -covers with Artin-Schreier conductor equal to k is

$$\sim \left(\frac{1 - q^{-p^e+1}}{1 - q^{-p^e-1}} - \frac{1 - q^{-p^{e-1}+1}}{1 - q^{-p^{e-1}-1}} \right) q^{2k} = B \cdot q^{-p^{e-1}+1} \cdot q^{2k}$$

where $B = \frac{1 - q^{-2} - q^{-p^e+p^{e-1}} + q^{-p^e+p^{e-1}-2}}{(1 - q^{-p^e-1})(1 - q^{-p^{e-1}-1})}$.

Theorem 3.8. Let G be an abelian p -group. There exists an ordinary G -cover of $\mathbb{P}^1$ if and only if the exponent of G is p .

Proof. ($\Leftarrow$) Suppose G is an abelian group of exponent p . G is therefore an elementary abelian group and suppose that $g_1, \dots, g_r$ generate G . Let P be any place with $\deg \mathfrak{p} > r$. Then, it is easy to see that there exists a $\varphi_{\mathfrak{p}} : U_{\mathfrak{p}}^1/U_{\mathfrak{p}}^2 \rightarrow G$ such that it is a surjection. For all other places $\mathfrak{q} \neq \mathfrak{p}$, let $\varphi_{\mathfrak{q}} : U_{\mathfrak{q}}^1/U_{\mathfrak{q}}^2 \rightarrow G$ be the constant map to the identity.

($\implies$) If G has exponent $> p$, then there is no non-constant map $\varphi_{\mathfrak{p}} : U_{\mathfrak{p}}^1/U_{\mathfrak{p}}^2 \rightarrow G$. Hence, all ramified places must have a last jump > 2 and thus by Corollary 3.3, φ defines a G -cover which is not ordinary. $\square$

The following theorem quantifies the failure of a G -cover to be ordinary.

Theorem 3.9. Let $G = \prod_{e=1}^r (\mathbb{Z}/p^e\mathbb{Z})^{m_e}$. If C is a curve over $\mathbb{F}_q$ that is a G -cover of $\mathbb{P}^1$, then the p -corank is bounded below by

$$g(C) - f(C) \geq \sum_{e=1}^r m_e \frac{p(p^{e-1} - 1)(p^e - 1)}{p + 1}.$$

Proof. Firstly, for some sufficiently large q , there is a curve that attains the claimed p -corank.... write some more details

Let $\varphi : \mathbb{A}_K/K^\times \rightarrow G$ be a surjection. We are claiming that the minimal p -corank is attained when the image of $\varphi_{\mathfrak{p}}$ is a cyclic direct summand of G for all ramified places $\mathfrak{p}$.

Suppose not. If $\varphi_{\mathfrak{p}}$ maps to some copy of $\mathbb{Z}/p^e\mathbb{Z}$ but not surjectively, there exists some distinct place $\mathfrak{p}'$ such that $\varphi_{\mathfrak{p}'}$ surjects onto it. Thus, we may replace $\varphi_{\mathfrak{p}}$ with the constant map to the identity and we would obtain a smaller p -corank whilst φ being still surjective. Thus, for some place $\mathfrak{p}$, the image of $\varphi_{\mathfrak{p}}$ is some non-cyclic subgroup of G . Without loss of generality, it therefore suffices to show that the p -corank is higher when a surjective φ is ramified at only one place $\mathfrak{p}$ than the p -corank of

We start with an easy observation. Let $l_1 \leq l_2 \leq \dots \leq l_k$ and $l'_1 \leq l'_2 \leq \dots \leq l'_k$ be the lower numbering ramification breaks at $\mathfrak{p}$ and $\mathfrak{p}'$ respectively for some surjective $\varphi, \varphi' : \mathbb{A}_K/K^\times \rightarrow G$. If $l_i \leq l'_i$ for all i , and $\deg \mathfrak{p} = \deg \mathfrak{p}'$ then the contribution to the p -corank $\sum_{i \geq 2} (\#G_i - 1) \cdot \deg \mathfrak{p}$ is higher for φ' at $\mathfrak{p}'$ than for φ at $\mathfrak{p}$. Another simple observation that we make is that we want $\deg \mathfrak{p} = 1$ to minimize the contribution to the p -corank. We can do this by assuming q is sufficiently large.

In the case where we only have one ramified place $\mathfrak{p}$ with $\varphi_{\mathfrak{p}} : U_{\mathfrak{p}}^1 \twoheadrightarrow G$, if we are to minimize the upper numbering ramification breaks, using 2.3 we can see that these occur at $\underbrace{1 \leq 1 \leq \dots \leq 1}_{\sum_{i=1}^r m_i = \log_p |G[p]|} \leq \underbrace{p \leq p \leq \dots \leq p}_{\sum_{i=2}^r m_i = \log_p |G[p^2]|} \leq \dots \leq \underbrace{p^r \leq p^r \leq \dots \leq p^r}_{m_r = |G[p^r]|}$.

Computing the p -corank formula for this case, we obtain

$$\sum_{l=1}^r (p^l - p^{l-1}) \cdot (|G|/|G[p^l]|) \cdot (|G[p^l]| - 1).$$

The $p^l - p^{l-1}$ refers to the difference in the two breaks in the upper numbering. The factor $|G|/|G[p^l]|$ is the scaling factor to make it count the number of indices with lower numbering where the ramification groups are isomorphic to $G[p^l]$. The last thing to verify is the inequalities between the two different p -coranks. $\square$

Lemma 3.10. Let $G = \oplus_{e=1}^r (\mathbb{Z}/p^e\mathbb{Z})^{m_e}$. Then,

$$\sum_e^r m_e \frac{p(p^{e-1} - 1)(p^e - 1)}{p + 1} \leq \sum_{e=1}^r (p^e - p^{e-1}) \cdot \frac{|G|}{|G[p^e]|} \cdot (|G[p^e]| - 1)$$

Proof. Let $s = \sum m_e e$ and $t = \sum m_e$. Without loss of generality, assume $t \geq 2$.

We first treat the case $p > 2$. The lower hand side can be upper bounded by L.H.S. $\leq \sum m_e p^{2e-1} \leq t p^{2r-1}$. The right hand side can be lower bounded by taking the summand for $e = r$.

$$\text{R.H.S.} \geq p^r(1 - 1/p) \cdot p^s(1 - 1/p^{rm_r}) \geq p^r(1 - 1/p) \cdot p^s(1 - 1/p) = p^{r+s}(1 - 1/p)^2.$$

Writing $s = \sum m_e e = \sum m_e(e - 1) + \sum m_e \geq r - 1 + t$. Thus, R.H.S. $\geq p^{2r-1+t}(1 - 1/p)^2$. So, it suffices to show that $p^t(1 - 1/p)^2 \geq t$. If we consider the real-valued function $g(t) = p^t(1 - 1/p)^2 - t$, it can be shown that $g'(t) > 0$ for $t \geq 2, p \geq 3$ and since $g(2) = (p-1)^2 - 2 > 0$, $g(t)$ is positive for all $t \geq 2, p \geq 3$. This shows that the lemma is true for $p \geq 3$ and $\sum m_e \geq 2$.

We now consider the $p = 2$ case. We can bound the left hand side as follows; L.H.S. $\leq t \frac{(2^r-2)(2^r-1)}{3} \leq \frac{1}{3} t 2^{2r}$. As for the right hand side, we take the summands for $e = r$ and $e = r - 1$.

$$\text{R.H.S.} \geq 2^{r-1} \cdot |G| \cdot (1 - 1/|G[p^r]|) + 2^{r-2} \cdot |G| \cdot (1 - 1/|G[p^{r-1}]|) \geq 2^{r-1} \cdot 2^s \cdot \frac{3}{4} \geq 2^{2r+t-2} \cdot \frac{3}{4}.$$

Now, it suffices to show that $1 \leq \frac{9}{16} \frac{2^t}{t}$. It is not difficult to check that for $t \geq 2$, this is true. $\square$

4 Elementary abelian extensions

Consider the elementary abelian group $G = (\mathbb{Z}/p\mathbb{Z})^m$. Here we study the distribution of ordinary G -covers of $\mathbb{P}^1$ defined over $\mathbb{F}_q$. We start with a simple observation in linear algebra and point counts on Grassmanians over finite fields.

Lemma 4.1. The number of group homomorphisms $f : (\mathbb{Z}/p\mathbb{Z})^{m_1} \rightarrow (\mathbb{Z}/p\mathbb{Z})^{m_2}$ is $p^{m_1 m_2}$.

Proof. Count the number of elements in $\text{Mat}_{m_1, m_2}(\mathbb{F}_p)$. $\square$

Lemma 4.2. The number of d -dimensional subspaces of $(\mathbb{Z}/p\mathbb{Z})^m$ is

$$p^{(m-d)d} \prod_{i=0}^{d-1} (1 - p^{i-m})(1 - p^{i-d})^{-1}.$$

Proof. There are $\prod_{i=0}^{d-1} (p^m - p^i)$ ways to choose d -linearly independent vectors in $(\mathbb{Z}/p\mathbb{Z})^m$. The map $\{\text{ordered } d \text{ linearly independent vectors in } (\mathbb{Z}/p\mathbb{Z})^m\} \rightarrow \text{Gr}(d, m), \mathcal{B} \mapsto \text{span}(\mathcal{B})$ is N -to-1 where $N = |\text{GL}(d, \mathbb{Z}/p\mathbb{Z})| = \prod_{i=0}^{d-1} (p^d - p^i)$. $\square$

Example. The number of 1-dimensional subspace of $(\mathbb{Z}/p\mathbb{Z})^m$ is $(p^m - 1)/(p - 1)$. However, there is a (non-canonical) isomorphism between $\text{Gr}(1, m)$ and $\text{Gr}(m - 1, m)$ given by sending V to its orthogonal complement $V^\perp$, after a choice of inner product. The number of $m - 1$ -dimensional can be represented as $p^{(m-d)d} \prod_{i=0}^{m-1} (1 - p^{i-m})(1 - p^{i-m+1})^{-1}$ which is in fact equal to $(p^m - 1)/(p - 1)$.

The next theorem follows from the proof of Theorem 3.4 and Lemma 4.1.

Theorem 4.3. Let $K = \mathbb{F}_q(t)$. A positive proportion of $(\mathbb{Z}/p\mathbb{Z})^m$ -covers of $\mathbb{P}^1$ are ordinary when ordered by Artin-Schreier conductor. That is to say,

$$\lim_{n \rightarrow \infty} \frac{\#\{C \text{ connected curve}/\mathbb{F}_q : \text{Gal}(k(C)/K) \cong (\mathbb{Z}/p\mathbb{Z})^m, C \text{ is ordinary, } \text{asc}(k(C)/K) = n\}}{\#\{C \text{ connected curve}/\mathbb{F}_q : \text{Gal}(k(C)/K) \cong (\mathbb{Z}/p\mathbb{Z})^m, \text{asc}(k(C)/K) = n\}} > 0$$

Proof. Define the generating function counting the number of isomorphism classes of (possibly disconnected) ordinary $(\mathbb{Z}/p\mathbb{Z})^m$ -covers of $\mathbb{P}_{\mathbb{F}_q}^1$ by the Artin-Schreier conductor.

$$F_{K, (\mathbb{Z}/p\mathbb{Z})^m, \text{ord}} := \frac{1}{|G|} \sum_{k \geq 0} \#\{C = \coprod C_i \text{ defined over } \mathbb{F}_q : \text{Gal}(k(C)/K) \cong (\mathbb{Z}/p\mathbb{Z})^m, \\ C \text{ is ordinary, } \text{asc}(k(C)/k) = n\} \cdot X^n.$$

Similar to 3.4, we know that $G^i(\mathfrak{P}/\mathfrak{p}) = 1$ for all $i \geq 2$ and for every ramified place $\mathfrak{p}$. The number of maps $U_{\mathfrak{p}}^1/U_{\mathfrak{p}}^2 \rightarrow (\mathbb{Z}/p^m\mathbb{Z})$ is p^{md} where $q^{\deg \mathfrak{p}} = p^d$. So, it is not hard to see that the generating function $F_{K, (\mathbb{Z}/p\mathbb{Z})^m, \text{ord}}(X)$ admits an Euler product with local factors being $1 + q^{m \deg \mathfrak{p}} X^{\deg \mathfrak{p}}$. Using the standard identity, $\zeta_{\mathbb{P}^1}(X) = \prod_P (1 - X)^{-1} = \frac{1}{(1 - qX)(1 - X)}$, we can check obtain that the generating function $F_{K, (\mathbb{Z}/p\mathbb{Z})^m, \text{ord}}(X)$ can be expressed as a rational function $\frac{(1 + q^m X)(1 - q^{2m} X^2)}{1 - q^{m+1} X}$. In particular, the number of (possibly disconnected) ordinary $(\mathbb{Z}/p\mathbb{Z})^m$ -covers of $\mathbb{P}^1$ with Artin-Schreier conductor equal to n is exactly equal to $Aq^{(m+1)n}$ where $A = (1 + q^{-1} - q^{-2} - q^{-3})$.

For n sufficiently large, using inclusion-exclusion principle and 4.2, we can say that the connected ordinary $(\mathbb{Z}/p\mathbb{Z})^m$ -covers of $\mathbb{P}^1$ with Artin-Schreier conductor equal to n is exactly

$$A \left(\sum_{d=0}^m (-1)^d q^{(m+1)n} p^{(m-d)d} \prod_{i=0}^{d-1} (1 - p^{i-m})(1 - p^{i-d})^{-1} \right) \\ = A(q^{(m+1)n} - q^{mn} \frac{(p^m - 1)}{p - 1} + q^{(m-1)n} \frac{(p^m - 1)(p^m - p)}{(p^2 - 1)(p^2 - p)} - \dots)$$

By [Gun26, 4.3], the generating function counting the (possibly disconnected) $(\mathbb{Z}/p\mathbb{Z})^m$ -covers of $\mathbb{P}_{\mathbb{F}_q}^1$ by the Artin-Schreier conductor is

$$F_{K, \text{asc}}(X) = \frac{(1 - X)(1 - qX)(1 - q^{pm} X^p)(1 - q^{pm+1} X^p)}{(1 - q^m X)(1 - q^{m+1} X)(1 - q^{(p-1)m} X^p)(1 - q^{(p-1)m+1} X^p)}.$$

The pole of $F_{K,\text{asc}}(X)$ closest to the origin is at $X = q^{-m-1}$. Therefore, we get that

$$\#\{C/\mathbb{F}_q : \text{Gal}(k(C)/K) \cong (\mathbb{Z}/p\mathbb{Z})^m, \text{asc}(k(C)/K) = n\} \sim Bq^{n(m+1)}$$

where $B = \frac{(1-q^{-r-1})(1-q^{-r})(1-q^{-p})(1-q^{-p+1})}{(1-q^{-1})(1-q^{-r-p})(1-q^{-r-p+1})}$.

□

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