Research Statement

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My research is mainly focused on understanding arithmetic statistics of wild ramification (both in the local and global setting) and applications to p-ranks of Galois covers.

Local counting problem in the wild case

A famous result of Kranser roughtly states that for a p-adic field K if the p-adic distance between two elements $\alpha, \beta \in \overline{K}$ is small compared to its Galois conjugates, the field extension $K(\alpha)$ and $K(\beta)$ are equal. This has the following implication; there are only *finitely many* degree n extensions of a p-adic field K. This fails for local function fields when the degree of extension is divisible by its characteristic. For instance, Artin-Schreier theory gives us that there are infinitely many $\mathbb{Z}/p\mathbb{Z}$-extensions of the characteristic p field $\mathbb{F}_q(\!(t)\!)$.

In order to get a finite count on the number of degree n extensions (where p divides n), we need to bound invariants like the *discriminant* or *last jump* which measure how ramified the extension is. In arithmetic statistics, we are often concerned with counting the number of field extensions with bounded invariants. For instance, Malle's conjecture on the number of Galois extensions of a number field with a prescribed Galois group G and discriminant bounded by X has been influential in the field of arithmetic statistics. However, unlike Malle's conjecture we do not have a conjecture on the distribution of these local function fields.

- (Discriminant) Lagemann[Lag12][Lag15] obtained lower and upper bounds for the abelian case.
- (Discriminant) Potthast [Pot26] obtains exact asymptotics for the elementary abelian case (i.e. $G = (C_p)^n$).
- (Last jump) Klüners and Müller [KM20] obtained exact asymptotics for the abelian case.
- (Last jump) Gundlach and Seguin [GS25] obtained uniform upper bounds for p-groups of nilpotency class 2.

Essentially, in the non-abelian case very little is known about these counts of local fields with bounded last jump or bounded discriminant. Gundlach and Seguin observed that works of Abrashkin[Abr95] gave a Lie-algebraic parametrization of G-extensions. The particularly useful thing about Abrashkin's parametrization is an explicit description of the ramifcation filtration when G is a p-group of nilpotency class $n < p$. The ramification filtration is a filtration on the Galois group of local fields(i.e. inertia groups), and information about this filtration can be used to compute invariants like the discriminant and last jump.

Theorem 1 (N.)**.** Let G be a p-group of maximal class, nilpotency class $n < p$ and exponent p. Then, there is a uniform lower bound on the number of etale G-extensions of $\mathbb{F}_q(\!(t)\!)$ with bounded last jump. More precisely,

$$\#\{L : \mathrm{Gal}(L/\mathbb{F}_q(\!(t)\!)) \cong G, \mathrm{lastjump}(L) \leq N\} \gg q^{(1-1/p)(0.6449N)}.$$

As with many results in arithmetic statistics, there is a signifcant group-theoretic input into proving this result. Group theorists have been studying called p-groups of maximal class (groups of order p^{n+1} and nilpotency class n). Some concrete examples of such p-groups include the Heisenberg group $\mathcal{H}_3(\mathbb{F}_p)$ and for $n < p$, the group $\mathbb{F}_p^n \rtimes \mathbb{F}_p \subset \mathrm{Affine}(\mathbb{F}_p^n)$ generated by all translations and a linear map of order p. The latter has nilpotency class $n < p$ and is therefore a family of p-groups not covered in previous works. The condition of the exponent being p may seem quite restrictive but in the case of maximal class p-groups, we know that all the subgroup $\gamma_2(G) = [G, G]$ of index p^2 is already of index p [LGM02][Bla58] and we can recover a similar bound with minimal correction for the case of exponent being p^2. I have been able to obtain stronger versions of the theorem if we have further knowledge about a particular property called the *degree of commutativity* which is defined as $\max\{d : [\gamma_i, \gamma_j] \subset \gamma_{i+j+d}\}$. An even better understanding of the nilpotency class of subgroups in the lower central series should allow us to get better lower bounds on these counts.

Non-integral last jumps

The Hasse-Arf Theorem says that if G is an abelian group, all the jumps in the ramification filtration in upper numbering are integers.

For the Heisenberg group $\mathcal{H}_3(\mathbb{F}_p)$, it is known that the last jump need not be an integer [EK23, 5.3]. I was able to show that for an integer N coprime to p,

$$\#\{L : \mathrm{Gal}(L/\mathbb{F}_q(\!(t)\!)) \cong \mathcal{H}_3(\mathbb{F}_p), \mathrm{lastjump}(L/\mathbb{F}_q(\!(t)\!)) = N\} \gg q^{2N(1-1/p)}.$$

On the other hand, let us consider non-integral last jumps. Let a, b be two positive integers such that $a, b \not\equiv 0 \pmod{p}$ and $a + b \not\equiv 0 \pmod{p}$. Then, I was able to show that the number of G-extensions with ramification breaks, $a < b < b + a/p$ can be bounded below by

$$\#\{L : \mathrm{Gal}(L/\mathbb{F}_q(\!(t)\!) \cong \mathcal{H}_3(\mathbb{F}_p), U_{L/K} = \{a, b, b + a/p\}\} \gg q^{(2b+a)(1-1/p)}.$$

This has the counterintuitive implication that the non-integral last jumps seem to be more abundant than integral last jumps when ordered by the last jump. For the Heisenberg group $\mathcal{H}_3(\mathbb{F}_p)$, the lower bound given here gets very close to the upper bound given by Gundlach and Seguin[GS25].

Wild ramifications in global function fields: wild versions of Malle's conjecture

Wild counting problems are also not properly understood in the global setting as well. The aforementioned works of Potthast[Pot26] and Gundlach, Seguin[GS25] covered the elementary-abelian case for discriminants and nilpotency class two case for the last jump, respectively.

In the tame case (the case where $(\mathrm{char}(K), |G|) = 1$), an influential conjecture in arithmetic statistics is a famous conjecture of Malle.

Conjecture (Malle's conjecture)**.** Let K be a number field. Then, there are constants $a = a(G), b = b(G, K)$ such that

$$\#\{L : \mathrm{Gal}(L/K) \cong G, \mathrm{disc}(L/K) \leq X\} \sim X^a (\log X)^b.$$

Originally conjectured for number fields, whilst very far from being fully resolved, the conjecture is known to be true for many families of groups[Wan21][ALOWW26][Klü12]. The conjecture has gone through multiple refinements and revisions[Klü05][Tür15][Wan25] but it is widely believed that the a-constant is correct. The tame Malle conjecture for the function field setting has also been extensively studied using topology of Hurwitz spaces[EVW16][ETW23][LL25].

However, the conjecture is known to be incorrect in the wild setting; even the a-constant appearing in the asymptotic is known to be incorrect for the elementary abelian case. An ambitious goal for us would be to have a conjectural distribution on such wild extensions counted by discriminant/last jump. Unlike the tame case, the moduli spaces that parametrize these wild covers behave fundamentally different from the Hurwitz spaces over $\mathbb{C}$. Currently, we do not have enough results for such counting problems for non-abelian p-groups to make such a conjectural statement.

Arithmetic Invariants: p-ranks of curves

Understanding the ramification filtration and its breaks also has applications to understanding invariants of curves. Relating back to what we said in the previous section, the genus of a curve is close related with the discriminant of its function field; these two invariants are both computed from the degree of the ramification divisor. In the tame case, the contribution to the ramification divisor comes from only the inertia group whereas, we need to keep track of all subgroups in the ramification filtration for the wild case. Counting G-covers by genus is essentially the same thing as counting G-extensions by discriminant. We can ask more refined questions like how many G-covers are there of a given a-number or p-rank? These are invariants of curves which can be used to study the group scheme $J(C)[p]$, the p-torsion of the Jacobian of a curve C. Determining the exact isomorphism type of this finite flat group scheme is, generally speaking difficult. There are various invariants to study these group schemes such as the a-number, p-rank and more refined invariants like the Ekedahl-Oort type. The p-rank denoted f_C can be readily computed when $C \to \mathbb{P}^1$ is a G-cover for a p-group G. If the p-rank agrees with the genus g_C, a curve is said to be ordinary, similar to the more familiar dichotomy of elliptic curves being ordinary versus supersingular. For a general wild cover, this ordinarity condition is equivalent to saying that at all local places, the last jump of the ramification filtration is at 1. An earlier work of Sankar[San20] showed that for $p \neq 2$, the proportion of Artin-Schreier covers($\mathbb{Z}/p\mathbb{Z}$-covers) that are ordinary is zero when ordered by genus. Sankar's work was seen as a counterexample to some heuristics on the arithmetic statistics of p-ranks and a-numbers [CEZB13] . This arithmetic statistical fact is related to the work of Pries and Zhu [PZ12]. In particular, they showed that $\mathcal{AS}_g$, the moduli space of Artin-Schreier covers is usually not irreducible and the number of irreducible component grows as the genus grows. The ordinary locus is a single component of this moduli space and thus, it gets dominated by the growing number of irreducible components.

A recent result of Gundlach[Gun26] showed that generating functions where the coefficients come from counting G-covers ordered by last jump (where G is an abelian p-group) is a rational function.

Theorem 2 (N.)**.** The generating function counting the number of ordinary Artin-Schreier

covers defined over $\mathbb{F}_q$ ordered by last jump has a rational expression. That is to say,

$$\frac{1}{p}\sum_{n\geq 0}\#\{C\to\mathbb{P}^1,\ \text{ordinary Artin-Schreier cover}/\mathbb{F}_q, \text{lastjump}(L/\mathbb{F}_q(t))=n\}\cdot X^n=\frac{(1+qX)(1-q^3X^2)}{1-q^2X}.$$

As a corollary, unlike Sankar's result [San20], a positive proprtion of Artin-Schreier covers are ordinary when ordered by last jump.

Corollary (N.)**.** Let $k=\mathbb{F}_q$ be a finite field.

$$\lim_{X\to\infty}\frac{\#\{C/k \text{ is an Artin-Schreier cover of } \mathbb{P}^1 \text{ that is ordinary}, \text{lastjump}(k(C)/k(\mathbb{P}^1)\leq X\}}{\#\{\#\{C/k \text{ is an Artin-Schreier cover of } \mathbb{P}^1, \text{lastjump}(k(C)/k(\mathbb{P}^1)\leq X\}}=a(q)>0$$

I was able to say for a given abelian p-group G, what is the smallest attainable p-corank (which we define to be the difference between genus and p-rank, g_C-f_C). I was also able to show analogous similar statements; we have a rationality result for the generating function for $\mathbb{Z}/p^e\mathbb{Z}$-covers of $\mathbb{P}^1$ ordered by last jump and a positive proportion of $\mathbb{Z}/p^e$-covers attain the lower bound on the p-corank when counted by last jump.

These rationality results strongly hint towards a moduli space interpretation with a stratification by the p-rank. If $\mathcal{X}$ is a moduli space that parametrizes an arithmetic object of interest (e.g. G-cover of $\mathbb{P}^1$), taking as input the trace of Frobenius acting on the cohomology groups, we can compute $\#\mathcal{X}(\mathbb{F}_q)$ by the Grothendieck-Lefschetz trace formula. If these point counts behave nicely (such as the generating function being rational), it is ususally because there is an homological stability of the underlying moduli space. From the perspective of arithmetic statistics, more work needs to be done in constructing these moduli space of wild covers especially ones with a given last jump. Even understanding the dimensions of these moduli spaces can be useful and needs to be explored more. More ambitiously, we need to develop a framework similar to [EVW16] for the tame case where we can establish homological stability results for these wild Hurwitz spaces.

We may also study which ordinarity using Abrashkin's nilpotent Artin-Schreier theory. The following result describes which G-covers can be ordinary.

Theorem 3 (N.)**.** Let G be an a p-group of nilpotency class $n<p$. There is an ordinary G-cover $C\to\mathbb{P}^1$ if and only if G is of exponent p.

This should not extend to arbitrary p-groups and the nilpotency class assumption is necessary. A crucial piece of Abrashkin's theory is the use of Lazard correspondence. A $\mathbb{Z}_p$-Lie algebra of finite cardinality can be viewed as a p-group under the Baker-Campbell-Hausdorff formula, a p-adic version of the more familiar Lie correspondence between real Lie algebras and simply connected real Lie groups. In rough terms, the Lazard correspondence can be thought of as saying that p-groups of nilpotency class $n<p$ can be *linearized*. A particularly nice thing about being able to linearize such groups is that having a set of generators all of order p is equivalent to the group having exponent p. In general, this need not be true; there are p-groups of exponent p^i for $i\geq 2$ with a set of generators of order p.

Abrashkin's nilpotent Artin-Schreier theory and other ways of computing the ramification filtration

Whilst Abrashkin's nilpotent Artin-Schreier theory gives explicit description of the ramification filtration, it can be difficult to work with.

Theorem (Abrashkin)**.** Let L be a finite $\mathbb{Z}_p$-Lie algebra of nilpotency class $n < p$ and $G(L)$ be the corresponding finite p-group. $G(L)$-extensions of $K = \mathbb{F}_q(\!(t)\!)$ are in bijection with $L \otimes_{\mathbb{Z}_p} W(K)//_{W_K}$. Given $x = \sum_{i \leq n} D_i t^{-i} \in L \otimes_{\mathbb{Z}_p} W(K)//_{W(K)}$, the ramification filtration with upper numbering satisfies $G^{(v)} = G(L(v))$ where $L(v)$ is the smallest σ-invariant Lie algebra generated by $\{\mathcal{F}_{-N}(\gamma) : \gamma \geq v\}$. These $\mathcal{F}_{-N}(\gamma)$ are defined as

$$\mathcal{F}_{-N}(\gamma) := \sum a_1 p^{n_1} [\sigma^{n_1}(D_{a_1}), [\sigma^{n_2}(D_{a_2}), ...[\sigma^{n_{k-1}}(D_{a_{k-1}}), \sigma^{n_k}(D_{a_k})]...]$$

where the sum is defined over all $(\underline{a}, \underline{n})$ with $\gamma = a_1 p^{n_1} + ... + a_k p^{n_k}$, $k < p$ and $n_1 \geq n_2 \geq ... \geq n_k \geq -N$.

One way to simplify the problem is to understand conditions that guarantee when the nested Lie bracket appearing in Abrashkin's theorem is zero and how often this happens. This is the main idea of Theorem 1. In [GS26a], Gundlach and Seguin studied problems such as counting matrices $A \in M_n(\mathbb{F}_q)$ which commute with their Frobenii $\sigma(A)$ motivated by understanding the nilpotency class two case. Gundlach and Seguin asked if we can study the zero sets of $\mathcal{F}_{-N}(\gamma)$. Because of the Frobenius acting on the coordinates and the presence of the Lie bracket, some generalizations of a variety is needed to make this into a precise geometric problem.

Gundlach and Seguin [GS26b] showed a way to compute the last jump of nilpotency two p-groups bypassing Abrashkin's theory and by understanding embedding problems; they were able to re-obtain some of their earlier results in [GS25] but now in characteristic two as well. Imai [Ima25][Ima26] was able to compute the ramification filtration of Galois groups isomorphic to p-groups of some matrix groups with nilpotency class p; the nilpotency class is too high for Abrashkin's techniques to apply.

Abrashkin's theory uses the classical Lazard correspondence but one can ask if generalizations of the Lazard correspondence can be useful. A mild generalization observed by Naik[Nai13] was that since the group axioms such as associativity, only requires at most three elements to describe it, it suffices to consider subgroups generated by three elements to satisfy the required nilpotency condition. The 3-local Lazard correspondence says that if G is a p-group where every subgroup generated by three elements has nilpotency class $< p$ then there is a corresponding $\mathbb{Z}_p$-Lie algebra. Can the theory be extended to some of these bigger class of p-groups?

Summary: Future Directions

As discussed, the area of wild ramification has a lot of things we have yet to understand and lacking heuristic arguments that motivate what we should expect.

- Better understanding on how to apply Abrashkin's techniques to study arithmetic statistics of wild ramification.

- Developing other techniques to compute the ramification filtration
- Numerical improvements on lower bounds that can be obtained from finer knowledge on p-groups, especially properties of subgroups in the lower central series.
- Understanding what properties of p-groups govern these statistics and coming up with a conjectural distribution of these stastics.

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