Group Theory **NOTES 3**
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THE FITTING AND FRATTINI SUBGROUPS

Suppose that $N_1, N_2, \ldots, N_r$ are normal subgroups of some finite group G. Because the N_i are normal, their product $M = \prod N_i$ is clearly a normal subgroup of G and the order in which the factors are multiplied is irrelevant. Also, by the standard formula for the order of the product of two subgroups, we see that the order of M is at most $\prod |N_i|$. Of course, each subgroup N_i is contained in M, so by Lagrange's theorem, we see that $|N_i|$ divides $|M|$ for each i. It follows that if the orders of the subgroups N_i are pairwise coprime, then $\prod |N_i|$ divides $|M|$, and thus equality must hold and we have $|M| = \prod |N_i|$.

We apply the observations of the previous paragraph to the normal subgroups $\mathbf{O}_p(G)$ of G and we observe that these subgroups have pairwise coprime orders. (Note that we need consider only primes that divide $|G|$.) We define the **Fitting subgroup** $\mathbf{F}(G)$ of the finite group G to be the product of the normal subgroups $\mathbf{O}_p(G)$ for all primes p. It follows that $\mathbf{F}(G)$ is normal in G (and in fact is characteristic) and its order is $\prod |\mathbf{O}_p(G)|$. From this order formula we see that for each prime p, the subgroup $\mathbf{O}_p(G)$ is actually a Sylow p-subgroup of $\mathbf{F}(G)$. Thus $\mathbf{F}(G)$ has a normal Sylow subgroup for each prime, and hence it is nilpotent.

In fact, $\mathbf{F}(G)$ is the unique largest normal nilpotent subgroup of G. To see this, we suppose that $N \triangleleft G$ is nilpotent, and we show that $N \subseteq \mathbf{F}(G)$. Since any finite group is generated by a collection of Sylow subgroups, one for each prime divisor of its order, it suffices to prove that $\mathbf{F}(G)$ contains a full Sylow p-subgroup of N for each prime p. Consider, therefore, $P \in \mathrm{Syl}_p(N)$. Since N is nilpotent, we have $P \triangleleft N$, and thus P is characteristic in N and hence is normal in G. We conclude that $P \subseteq \mathbf{O}_p(G) \subseteq \mathbf{F}(G)$, as required.

A corollary that is not completely obvious from first principles is the following.

COROLLARY 1. *Let $M, N \triangleleft G$, where M and N are each nilpotent. Then MN is nilpotent.*

Proof. Each of M and N is contained in $\mathbf{F}(G)$ and thus MN is a subgroup of the nilpotent group $\mathbf{F}(G)$. It follows that MN is nilpotent. ∎

Another useful, but trivial consequence of the previous discussion is the following.

COROLLARY 2. *Let $N \triangleleft G$. Then $N \cap \mathbf{F}(G) = \mathbf{F}(N)$.*

Proof. Certainly, $\mathbf{F}(N)$ is nilpotent, and since it is characteristic in N, it is normal in G. It follows that $\mathbf{F}(N) \subseteq N \cap \mathbf{F}(G)$. Also, since $\mathbf{F}(G) \triangleleft G$, we know that $N \cap \mathbf{F}(G) \triangleleft N$. This normal subgroup of N is nilpotent, however, and thus it is contained in $\mathbf{F}(N)$. ∎

Observe that if G is a nonidentity solvable group, then G has a nontrivial abelian normal subgroup. Since an abelian group is certainly nilpotent, it follows that $\mathbf{F}(G) > 1$. The following theorem is a stronger version of the fact that nontrivial solvable groups have nontrivial Fitting subgroups.

THEOREM 3. *Let G be solvable. Then* $\mathbf{F}(G) \supseteq \mathbf{C}_G(\mathbf{F}(G))$.

In other words, the fitting subgroup of a nontrivial solvable group is not only nontrivial, it is even big enough to contain its own centralizer. This is a much stronger conclusion. The key lemma for the proof of this theorem is the following.

LEMMA 4. *Let $Z \subseteq \mathbf{Z}(G)$. Then* $\mathbf{F}(G/Z) = \mathbf{F}(G)/Z$.

Proof. We let $F/Z = \mathbf{F}(G/Z)$ and we show that actually, $F = \mathbf{F}(G)$. Since F/Z is nilpotent and Z is central in F, we see that F must be nilpotent. (This is proved twice in the notes on nilpotent groups.) We now know that $F \subseteq \mathbf{F}(G)$.

Of course $Z \subseteq \mathbf{F}(G)$, and we have that $\mathbf{F}(G)/Z$ is nilpotent since $\mathbf{F}(G)$ is nilpotent. Thus $\mathbf{F}(G)/Z \subseteq \mathbf{F}(G/Z) = F/Z$, and it follows that $\mathbf{F}(G) \subseteq F$. ∎

Proof of Theorem 3. Write $F = \mathbf{F}(G)$ and $C = \mathbf{C}_G(F)$, and note that $C \triangleleft G$, so that $F \cap C = \mathbf{F}(C)$ by Corollary 2. Also, $F \cap C \subseteq \mathbf{Z}(C)$ since C centralizes F. By Lemma 4, we have $\mathbf{F}(C/(F \cap C)) = \mathbf{F}(C)/(F \cap C) = (F \cap C)/(F \cap C)$, and this is trivial. Thus the solvable group $C/(F \cap C)$ has a trivial Fitting subgroup, and thus it must be a trivial group. In other words, $C = F \cap C$, and this is contained in F, as required. ∎

The **Frattini subgroup** of a group G, denoted $\Phi(G)$, is the intersection of all maximal subgroups of G. Of course, $\Phi(G)$ is characteristic, and hence normal in G, and as we will see, it is nilpotent. It follows that for any finite group G, we have $\Phi(G) \subseteq \mathbf{F}(G)$.

Actually $\Phi(G)$ has a property stronger than being nilpotent.

THEOREM 5. *Let $\Phi(G) \subseteq N \triangleleft G$ and suppose that $N/\Phi(G)$ is nilpotent. Then N is nilpotent.*

In Theorem 5, we can take $N = \Phi(G)$ so that $N/\Phi(G)$ is the trivial group, and hence is nilpotent. The theorem then tells us that $\Phi(G)$ is nilpotent, as we said earlier.

To prove Theorem 5 we need the familiar "Frattini argument", which we state without proof. (It is an immediate consequence of the Sylow C-Theorem, applied in the group N.)

LEMMA 6 (Frattini argument). *Let $N \triangleleft G$ and suppose that $P \in \mathrm{Syl}_p(N)$. Then* $G = \mathbf{N}_G(P)N$. ∎

Proof of Theorem 5. To show that N is nilpotent, we prove that each of its Sylow subgroups is normal. For this purpose, we let $P \in \mathrm{Syl}_p(N)$ and we note that $P\Phi(G)/\Phi(G)$ is a Sylow p-sugroup of $N/\Phi(G)$. But $N/\Phi(G)$ is assumed to be nilpotent, and thus its Sylow subgroups are normal and we have $P\Phi(G)/\Phi(G) \triangleleft N/\Phi(G)$. Thus, in fact, $P\Phi(G)/\Phi(G)$ is actually characteristic in $N/\Phi(G)$ and since $N/\Phi(G) \triangleleft G/\Phi(G)$, we deduce that $P\Phi(G)/\Phi(G) \triangleleft G/\Phi(G)$. By the correspondence theorem, this gives $P\Phi(G) \triangleleft G$.

Since P is Sylow in N, it is also Sylow in $P\Phi(G)$. Since the latter subgroup is normal in G, we can apply the Frattini argument to deduce that $\mathbf{N}_G(P)P\Phi(G) = G$. Since $P \subseteq \mathbf{N}_G(P)$, however, this yields $\mathbf{N}_G(P)\Phi(G) = G$. It follows from this that $\mathbf{N}_G(P) = G$. (Otherwise, $\mathbf{N}_G(P)$ would be contained in some maximal subgroup of G, which also contains $\Phi(G)$, and this would contradict the fact that $\mathbf{N}_G(P)\Phi(G) = G$.)

We now have $\mathbf{N}_G(P) = G$, and thus $P \triangleleft G$. In particular, $P \triangleleft N$ as desired. ∎

Note that Theorem 5 tells us that $\mathbf{F}(G/\Phi(G)) = \mathbf{F}(G)/\Phi(G)$.