Group Theory **NOTES 2**
I. M. Isaacs Fall 2002

REVIEW OF NILPOTENT GROUPS AND p-GROUPS

The basic lemma that we will need here is the following.

LEMMA 1. *Let P be a (finite) p-group and let $1 < N \triangleleft P$. Then $N \cap \mathbf{Z}(P) > 1$.*

In other words, every nonidentity normal subgroup of a p-group contains some nonidentity element of the center $\mathbf{Z}(P)$, and in particular, the center of a nontrivial p-group is never trivial. We will not give the details of the proof of Lemma 1; it is an immediate consequence of the FCP in the conjugation action of P on the subset $N - \{1\}$.

Given any group, we recursively define normal subgroups Z_i of G for all integers $i \geq 0$. First, we set $Z_0 = 1$ and $Z_1 = \mathbf{Z}(G)$. To define Z_2, consider the center $\mathbf{Z}(G/Z_1)$. Since this is a subgroup of G/Z_1 it follows by the "correspondence" theorem that it must have the form H/Z_1 for some uniquely defined subgroup H of G, and we define $Z_2 = H$. Observe that Z_2 is canonically defined, and so it is characteristic in G, and in particular it is normal. (The subgroup Z_2 is sometimes called the **second center** of G.) At the next step we define the subgroup $Z_3 \triangleleft G$ such that $Z_3/Z_2 = \mathbf{Z}(G/Z_2)$, and we continue this process to obtain a series of normal subgroups $Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_i \subseteq \cdots$ such that $Z_{i+1}/Z_i = \mathbf{Z}(G/Z_i)$ for every subscript $i \geq 0$. This series of normal subgroups is called the **upper central series** of G.

Now, suppose that G is a finite p-group. If $G > 1$, we know that $Z_1 = \mathbf{Z}(G)$ is nontrivial, and thus $Z_0 < Z_1$. Also, if $Z_1 < G$, then G/Z_1 is a nontrivial p-group, and so it has a nontrivial center, and this implies $Z_1 < Z_2$. Continuing like this, we see that whenever $Z_i < G$, we have $Z_i < Z_{i+1}$. It follows that if G is a finite p-group, then $Z_r = G$ for some subscript r. In other words, the upper central series "reaches the top".

In general, consider a finite series of normal subgroups of a group G of the form $1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = G$. Such a series is said to be a **central series** for G if $N_{i+1}/N_i \subseteq \mathbf{Z}(G/N_i)$ for subscripts i with $0 \leq i < r$. (Note the requirement that $N_r = G$. Thus the upper central series of G is not a central series for G unless it reaches the top, as in the case where G is a finite p-group. If the upper central series of G does reach the top, then, of course, it is a central series for G.)

DEFINITION. A group is **nilpotent** if it has a central series.

COROLLARY 2. *A finite p-group is nilpotent.*

THEOREM 3. *Let G be a finite group. The following are then equivalent.*

(i) *G is nilpotent.*
(ii) *If $H < G$, then $\mathbf{N}_G(H) > H$. ("Normalizers grow.")*
(iii) *Every maximal subgroup of G is normal.*
(iv) *Every Sylow subgroup of G is normal.*

We need an easy lemma.

LEMMA 4. *Assume that G is nontrivial and satisfies condition (iv) of Theorem 3. Then* $\mathbf{Z}(G) > 1$.

Proof. Let P be the Sylow p-subgroup of G, where p is a prime divisor of $|G|$. Then $P > 1$, and so $\mathbf{Z}(P) > 1$, and we show that $\mathbf{Z}(P) \subseteq \mathbf{Z}(G)$. Write $Z = \mathbf{Z}(P)$ and $C = \mathbf{C}_G(Z)$, so that our goal is to show that $C = G$. If this is false, we choose a prime divisor q of $|G : C|$, and we derive a contradiction. Since $Z = \mathbf{Z}(P)$, we have $P \subseteq C$, and thus $q \neq p$. Now let Q be a Sylow q-subgroup of G, and note that $Q \cap P = 1$ since $q \neq p$. Since P and Q are both normal in G, it follows that Q centralizes P, and thus Q centralizes Z, and so $Q \subseteq C$. This contradicts the fact that q divides $|G : C|$. ∎

Proof of Theorem 3. First, assume (i) so that G has a central series $1 = N_0 \subseteq \cdots \subseteq N_r = G$. Given $H < G$, there must be some subscript i such that $N_i \subseteq H$ but $N_{i+1} \not\subseteq H$. For notational simplicity, write $N = N_i$ and $M = N_{i+1}$. Then $M/N \subseteq \mathbf{Z}(G/N) \subseteq \mathbf{N}_{G/N}(H/N) = \mathbf{N}_G(H)/N$, where the last equality follows from the correspondence theorem. It follows that $M \subseteq \mathbf{N}_G(H)$. Since $M \not\subseteq H$ and $\mathbf{N}_G(H) \supseteq H$, it follows that $\mathbf{N}_G(H) > H$, proving (ii).

Now assume (ii) and let M be a maximal subgroup of G. Then $M < \mathbf{N}_G(M)$, and hence by the maximality of M, we have $\mathbf{N}_G(M) = G$, and thus $M \triangleleft G$. This proves (iii).

Now assume (iii) and let P be a Sylow subgroup of G. Assuming that P is not normal, we can choose a maximal subgroup M containing $\mathbf{N}_G(P)$. We know by (iii) that $M \triangleleft G$, and thus by the Frattini argument, we have $G = M\mathbf{N}_G(P)$. Since $\mathbf{N}_G(P) \subseteq M$, however, this yields $G = M$. This is a contradiction, and (iv) is proved.

Now assume (iv). We prove that G is nilpotent by showing that the upper central series of G reaches the top. For this purpose, it suffices to show that if $N \triangleleft G$ with $N < G$, then $\mathbf{Z}(G/N)$ is nontrivial. Note that the group G/N inherits from the group G the property that every Sylow subgroup is normal. (This follows easily from the fact that a surjective homomorphism of groups carries a Sylow p-subgroup to a Sylow p-subgroup.) Thus G/N satisfies (iv), and the result follows via Lemma 4. ∎

There is one more useful fact that we should mention.

LEMMA 5. *Let $Z \subseteq \mathbf{Z}(G)$ and suppose that G/Z is nilpotent. Then G is nilpotent.*

Proof. Let $1 = N_0/Z \subseteq N_1/Z \subseteq \cdots \subseteq N_r/Z = G/Z$ be a central series for G/Z. Then $1 \subseteq Z = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = G$ is easily seen to be a central series for G. ∎

If we assume in Lemma 5 that G is finite, then an alternative proof is available.

Second Proof of Lemma 5. Assuming that G is finite, it suffices by Theorem C to show that an arbitrary maximal subgroup M of G is normal in G. If $Z \subseteq M$, then M/Z is maximal in G/Z. In this case, we have $M/Z \triangleleft G/Z$ since G/Z is nilpotent, and thus $M \triangleleft G$ as required. In any case, $Z \subseteq \mathbf{Z}(G) \subseteq \mathbf{N}_G(M)$, and so if $Z \not\subseteq M$, then $\mathbf{N}_G(M) > M$, and hence $\mathbf{N}_G(M) = G$ in this case too. ∎