REVIEW OF GROUP ACTIONS

Let G be a group and Ω a set. (In this context, the members of Ω are often called "points".) Suppose that we have a particular rule that determines a unique point in Ω whenever we are given a point in Ω and an element of G. Specifically, if $\alpha \in \Omega$ and $g \in G$ are our input data, this rule (denoted by a dot) determines a unique point $\alpha{\cdot}g \in \Omega$. (More formally, "dot" can be thought of as a function from the Cartesian product $\Omega \times G$ to the set Ω.) In this situation, we say that G **acts** on Ω if the following two conditions hold for all points $\alpha \in \Omega$ and all elements $g, h \in G$.

(1) $\alpha{\cdot}1 = \alpha$.

(2) $(\alpha{\cdot}g){\cdot}h = \alpha{\cdot}(gh)$.

The basic example of a group action is the case where G is a subgroup of $\mathrm{Sym}(\Omega)$, the symmetric group on Ω, and dot is defined so that $\alpha{\cdot}g$ is just the result of applying g to α, so that $\alpha{\cdot}g = (\alpha)g$.

In general, if G acts on Ω, we can write θ_g to denote the function $\Omega \to \Omega$ induced by a group element g. In other words, the value of the function θ_g on the point α is precisely $\alpha{\cdot}g$. In symbols, we write $(\alpha)\theta_g = \alpha{\cdot}g$. The following is utterly routine:

LEMMA 1. *Let G act on Ω.*

(a) $\theta_g\theta_h = \theta_{gh}$ *for* $g, h \in G$.

(b) *The map* $g \mapsto \theta_g$ *is a group homomorphism from* G *into* $\mathrm{Sym}(\Omega)$.

(c) *The set* $N = \{g \in G \mid \theta_g = \text{identity map}\}$ *is a normal subgroup of* G.

(d) *If* N *is as above, then* G/N *is isomorphic to a subgroup of* $\mathrm{Sym}(\Omega)$.

We will not bother to write a detailed proof, but we make some remarks. The fact (asserted in (b)) that θ_g is a permutation of Ω is an immediate consequence of (a). Also, the normal subgroup described in (c) is called the **kernel** of the action. It is equal to the set of all of those elements $g \in G$ that fix all the points of Ω.

An extremely important example of a group action is the action on the right cosets of a subgroup. If $H \subseteq G$, let $\Omega = \{Hx \mid x \in G\}$, the set of right cosets of H in G. If Hx is any right coset and $g \in G$, we can "translate" Hx by g by multiplying each element of Hx on the right by g. This gives $(Hx)g = Hxg$, which is another right coset of H. We can thus define an action of G on Ω by setting $(Hx){\cdot}g = (Hx)g$, and it is a triviality to check that this is indeed an action.

Given a subgroup $H \subseteq G$, we define the **core** of H in G to be the subgroup obtained by intersecting all conjugates of H in G. Symbolically, we write $\mathrm{core}_G(H) = \bigcap H^x$, where x runs over G. We have the following.

THEOREM 2. *Let $H \subseteq G$. Then $\mathrm{core}_G(H)$ is the kernel of the natural action of G on the right cosets of H. Also, $\mathrm{core}_G(H)$ is a normal subgroup of G contained in H and it contains every other such normal subgroup.*

Proof. First, suppose that $N \lhd G$ and $N \subseteq H$. Then $N = N^x \subseteq H^x$ for all $x \in G$, and thus $N \subseteq \bigcap H^x = \mathrm{core}_G(H)$.

If we take M to be the kernel of the action of G, then of course $M \lhd G$ and we claim that also $M \subseteq H$. To see this, let $m \in M$ and note that $m \in Hm = (H)\cdot m = H$, where the last equality holds because m fixes the coset $H1 = H$. We conclude from the first paragraph that $M \subseteq \mathrm{core}_G(H)$

If $g \in \mathrm{core}_G(H)$ and $x \in G$, then $g \in H^x = x^{-1}Hx$, and thus $xg \in Hx$. It follows that $Hxg \subseteq HHx = Hx$, and thus $(Hx)\cdot g = Hx$ for every element $x \in G$. In other words, $\mathrm{core}_G(H)$ is contained in the kernel M of the action, and this completes the proof. ▮

The above result can be used to prove "nonsimplicity" theorems such as Corollary 4, below, which is often called the "$n!$-theorem".

COROLLARY 3. *Suppose $H \subseteq G$ and $|G : H| = n < \infty$. Let $N = \mathrm{core}_G(H)$. Then G/N is isomorphic to a subgroup of $\mathrm{Sym}(n)$, and thus $|G : N|$ divides $n!$.*

Proof. We know that N is the kernel of the action of G on $\Omega = \{Hx \mid x \in G\}$, which has n points. Also, by Lemma 1(d), we have that G/N is isomorphic to a subgroup of $\mathrm{Sym}(\Omega) \cong \mathrm{Sym}(n)$. ▮

COROLLARY 4. *Suppose $H \subseteq G$ with $|G : H| = n$, where $1 < n < \infty$. If $|G|$ does not divide $n!$, then $\mathrm{core}_G(H)$ is a nonidentity proper normal subgroup of G, and thus G is not simple.*

Proof. Write $N = \mathrm{core}_G(H)$. Then $N \lhd G$ and $N \subseteq H < G$. so that N is proper. If $N = 1$, then $|G| = |G : N|$ would divide $n!$. ▮

In addition to proving nonsimplicity theorems, group actions can also be used to count things. The key to this is the notion of an "orbit" of an action. Suppose G acts on Ω and let $\alpha \in \Omega$ be a point. Then the **orbit** of α in this action, denoted $\mathcal{O}_\alpha$, is the set of all points of the form $\alpha\cdot x$ as x runs over G. The following is nearly trivial.

LEMMA 5. *Let G act on Ω. If $\alpha \in \Omega$, then $\mathcal{O}_\alpha$ contains α and it is the only orbit of this action that contains α.*

Proof. Certainly $\alpha = \alpha\cdot 1 \in \mathcal{O}_\alpha$. If also $\alpha \in \mathcal{O}_\beta$ with $\beta \in \Omega$, we must show that $\mathcal{O}_\beta = \mathcal{O}_\alpha$. To see this, note that since $\alpha \in \mathcal{O}_\beta$, we can write $\alpha = \beta\cdot g$ for some element $g \in G$. Now if $\gamma \in \mathcal{O}_\alpha$, then for some element $x \in G$ we have $\gamma = \alpha\cdot x = (\beta\cdot g)\cdot x = \beta\cdot(gx)$, and thus γ lies in $\mathcal{O}_\beta$. This shows that if $\alpha \in \mathcal{O}_\beta$, then $\mathcal{O}_\alpha \subseteq \mathcal{O}_\beta$.

To prove the reverse containment, we observe that $\alpha\cdot g^{-1} = (\beta\cdot g)\cdot g^{-1} = \beta\cdot 1 = \beta$, and thus $\beta \in \mathcal{O}_\alpha$. By the result of the previous paragraph, we have $\mathcal{O}_\beta \subseteq \mathcal{O}_\alpha$. Thus $\mathcal{O}_\beta = \mathcal{O}_\alpha$, as required. ▮

As usual, suppose that G acts on Ω. Since every point of Ω lies in exactly one orbit, it follows that Ω is the disjoint union of all the orbits. If the whole set Ω consists of a single orbit, then we say that the action is **transitive**. (For example, the action of a group on the right cosets of any subgroup is transitive.) In general, if $\mathcal{O}$ is any orbit, then G acts transitively on $\mathcal{O}$.

If $\alpha \in \Omega$, we write $G_\alpha = \{x \in G \mid \alpha\cdot x = \alpha\}$. It is easy to see that G_α is a subgroup of G. It is called the **stabilizer** of α in G. For example, it is easy to check that in the action of G on the right cosets of a subgroup H, the stabilizer in G of the coset Hx is exactly the conjugate H^x.

THEOREM 6. *Let G act on Ω and let $\mathcal{O}$ be any orbit. Choose $\alpha \in \mathcal{O}$ and let $H = G_\alpha$ be the stabilizer of α in G. Then there is a bijection from the set of right cosets of H in G onto $\mathcal{O}$.*

Proof. First, note that $\mathcal{O} = \mathcal{O}_\alpha$, so that the members of $\mathcal{O}$ are exactly the points of the form $\alpha{\cdot}x$ for $x \in G$. Let $\mathcal{C} = \{Hx \mid x \in G\}$ be the set of right cosets of H in G. We wish to define a function $\varphi : \mathcal{C} \to \mathcal{O}$ by setting $\varphi(Hx) = \alpha{\cdot}x$, but we must check that this is well defined. If $Hx = Hy$, therefore, we must establish that $\alpha{\cdot}x = \alpha{\cdot}y$. But $y \in Hy = Hx$, and so we can write $y = hx$ for some element $h \in H$. Then $\alpha{\cdot}y = \alpha{\cdot}(hx) = (\alpha{\cdot}h){\cdot}x = \alpha{\cdot}x$, where the last equality follows because $h \in H = G_\alpha$, so that $\alpha{\cdot}h = \alpha$. We can now define φ as above, and our task is to show that it is both surjective and injective.

For surjectivity, assume that $\beta \in \mathcal{O}$. We must find $Hx \in \mathcal{C}$ such that $\varphi(Hx) = \beta$. We know, however, that $\beta = \alpha{\cdot}x$, for some element $x \in G$, and we have $Hx \in \mathcal{C}$. Then $\varphi(Hx) = \alpha{\cdot}x = \beta$, as required.

To prove that φ is injective, we suppose that $\varphi(Hx) = \varphi(Hy)$ and we show that $Hx = Hy$. We have $\alpha{\cdot}x = \alpha{\cdot}y$, and thus $\alpha = (\alpha{\cdot}x){\cdot}x^{-1} = (\alpha{\cdot}y){\cdot}x^{-1} = \alpha{\cdot}(yx^{-1})$. In other words, $yx^{-1} \in G_\alpha = H$. This yields $H = Hyx^{-1}$, and from this, we deduce that $Hx = Hy$, as required. ∎

COROLLARY 7. (The Fundamental Counting Principle.) *Let G act on Ω and suppose that $\mathcal{O}$ is a finite orbit. Then $|\mathcal{O}| = |G : H|$, where H is the stabilizer of any point in Ω. In particular, if G is finite, then $|\mathcal{O}| = |G|/|H|$ and this is a divisor of $|G|$.* ∎

There are a number of important immediate consequences of the FCP.

COROLLARY 8. *Let $x \in G$, where G is finite. Then the size of the conjugacy class of x in G is equal to $|G : \mathbf{C}_G(x)|$.*

Proof. The group G acts on itself by conjugation and the conjugacy class of x is exactly the orbit of x in this action. The stabilizer of x is $\mathbf{C}_G(x)$ and the result follows by the FCP. ∎

COROLLARY 9. *Let X be any subset of G, where G is finite. Then the number of distinct conjugates of X in G is equal to $|G : \mathbf{N}_G(X)|$.*

Proof. The group acts by conjugation on the set of all of its subsets. The orbit of X is the set of distinct conjugates of X and the stabilizer of X is $\mathbf{N}_G(X)$. ∎

COROLLARY 10. *Let $H, K \subseteq G$ be finite subgroups. Then $|HK| = |H||K|/|H \cap K|$.*

Proof. Let K act on the set of right cosets of H in G by right multiplication. Then $HK = \bigcup Hk$ as k runs over K, and thus HK is the union of all right cosets of H in the orbit containing H under this action of K. These cosets are disjoint and each has size equal to $|H|$, and so we see that $|HK| = n|H|$, where n is the size of the orbit. By the FCP, we know that $n = |K : D|$, where D is the stabilizer in K of H.

An element $k \in K$ lies in D precisely when $Hk = H$, and this happens precisely when $k \in H$. This shows that $D = H \cap K$, and thus $n = |K : D| = |K|/|H \cap K|$. The result now follows. ∎

The FCP also provides a tool for counting the number of orbits in an action. To explain this, we define the integer-valued function χ on G corresponding to its action on a finite set Ω. This function, the **permutation character** associated with the action, is defined by the formula $\chi(g) = |\{\alpha \in \Omega \mid \alpha{\cdot}g = \alpha\}|$. In other words, the value of the character at $g \in G$ is the number of fixed points of g in Ω. The following result, often incorrectly attributed to W. Burnside, says that the number of orbits of a finite group acting on a finite set is the average value of the associated permutation character.

THEOREM 11. *Let G act on Ω, where both G and Ω are finite, and suppose there are exactly n orbits in this action. Then*

$$n = \frac{1}{|G|} \sum_{g \in G} \chi(g) \,,$$

where χ is the associated permutation character.

Proof. Let $\mathcal{P}$ be the set of ordered pairs (g, α) with $g \in G$ and $\alpha \in \Omega$ and such that $\alpha{\cdot}g = \alpha$. Then for each element $g \in G$, there are precisely $\chi(g)$ members of $\mathcal{P}$ having g as first entry and we see that $|\mathcal{P}| = \sum \chi(g)$. Similarly, for $\alpha \in \Omega$, exactly $|G_\alpha|$ pairs in $\mathcal{P}$ have second entry α, and this yields $|\mathcal{P}| = \sum |G_\alpha|$, where this sum runs over $\alpha \in \Omega$. We now have

$$\frac{1}{|G|} \sum_{g \in G} \chi(g) = \frac{|\mathcal{P}|}{|G|} = \sum_{\alpha \in \Omega} \frac{|G_\alpha|}{|G|} = \sum_{\alpha \in \Omega} \frac{1}{|\mathcal{O}_\alpha|} \,,$$

where the last equality is a consequence of the FCP.

We compute the part of the final sum above contributed by the points lying in some particular orbit $\mathcal{O}$. For each point α in $\mathcal{O}$, we see that $\mathcal{O}_\alpha = \mathcal{O}$, and thus the contribution to the sum coming from α is $1/|\mathcal{O}|$. The total contribution from all points in $\mathcal{O}$ is thus $|\mathcal{O}|$ times $1/|\mathcal{O}|$. In other words, each orbit contributes a total of 1 to the sum on the right, which is therefore equal to n, as required. ∎

We close with the following Corollary.

COROLLARY 12 *Let $H < G$, where G is a finite group. Then there exists an element $g \in G$ that lies in no conjugate of H.*

Proof. Let G act on the set Ω of right cosets of H in G. Since the stabilizer of the coset (point) Hx is H^x, our problem is to find some element of G that fixes no point of Ω. We seek $g \in G$, in other words, with $\chi(g) = 0$.

Our action is transitive, and so the average value of the permutation character χ is 1. But $\chi(1) = |\Omega| = |G : H| > 1$ and χ has an above-average value at the identity 1. It follows that χ must have a below-average value at some element $g \in G$. Thus $\chi(g) < 1$, but since $\chi(g)$ is a non-negative integer, we deduce that $\chi(g) = 0$, as required. ∎

We mention that the conclusion of Corollary 12 can fail if G is an infinite group.