HOMEWORK 2, DUE MAR 1.

1. Let $K = \mathbf{Q}(\sqrt{d})$ be a real quadratic field with class group $\mathrm{Cl}(K)$ and narrow class group $\mathrm{Cl}^\infty(K)$. Show that the following are equivalent:

(a) The groups $\mathrm{Cl}(K)$ and $\mathrm{Cl}^\infty(K)$ coincide.

(b) The fundamental unit of K has norm -1.

Choose an example of d where the two groups do not coincide; find the two corresponding ray class fields (the Hilbert class field and the narrow class field).

2. Let $f(x) = x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 7 \in \mathbf{Z}[x]$.

(a) Show that $f \pmod p$ has repeated roots if p is 2 or 7.

By the "shape" of $f \pmod p$, we mean the multiset of degrees of its irreducible factors, so for instance if it is irreducible, its shape is $\{6\}$.

(b) For each partition π of 6, how many primes $p < 10^6, p \neq 2, 7$, have shape π.

Let G be the Galois group of f over $\mathbf{Q}$.

(c) Show that G is not a subgroup of A_6. Guess the order of G. Guess G up to isomorphism.

(d) Find all proper Galois subextensions of the splitting field of f.

[If you have not had much experience with computational algebra systems and need help getting started, please let me know.]