ECE/MATH 844: HOMEWORK 5, DUE DEC 8.

Please solve the following problems.

1. (a) Let E be an elliptic curve over $\mathbf{Q}$ with $j(E) \neq 0$. Show that E has a Weierstrass equation over $\overline{\mathbf{Q}}$ of the form

$$y^2 + axy + y = x^3 \quad (a \in \overline{\mathbf{Q}})$$

and that $(0,0)$ is a torsion point on E. What is its order?

(b) Let n be a positive integer and consider the problem of finding all nonzero integers u, v, w satisfying

$$(*) \quad u/v + v/w + w/u = n$$

Show that this amounts to finding $E_n(\mathbf{Q})$ for a certain elliptic curve E_n so long as $n \neq 3$.

(c) Show that the torsion subgroup of $E_n(\mathbf{Q})$ always has order divisible by 3. Find an n for which the torsion subgroup has order larger than 3.

(d) Show that the rank of $E_6(\mathbf{Q})$ is greater than 0. Find an upper bound for this rank. Find a non-torsion point in $E_6(\mathbf{Q})$ and the corresponding solution to $(*)$. Show that there are infinitely many solutions to $(*)$ with $n = 6$ and $u, v, w > 0$.

2. Suppose p and $p-2$ are both primes ($p \geq 5$). Let E be the elliptic curve over $\mathbf{Q}$ with equation $y^2 = x(x-2)(x-p)$.

(a) Using a computer algebra system such as MAGMA, investigate how the rank of $E(\mathbf{Q})$ varies with p. Give a conjecture, with cases depending on $p \pmod 8$ as to what the rank is.

(b) Prove this conjecture if $p \equiv 7 \pmod 8$. [Hint: show that many of the homogeneous spaces have no points by working in the reals or modulo powers of $2, p$, or $p-2$.]