MATH 844: HOMEWORK 11, DUE APR 20.

11. The integer equation $a^4 + ma^2b^2 + b^4 = c^2$ $(*)$ $(a, b) = 1, a, b > 0$ was studied by Fermat and Euler. A solution is called trivial if either $ab = 0$ or $a = b = 1$.

(a) Let E be the elliptic curve over $\mathbf{Q}$ given by $y^2 = x^3 + mx^2 + x$. Show that $(*)$ has a nontrivial solution if and only if the Mordell-Weil rank of E is nonzero.

(b) Euler showed that for $m = 14$, there are only trivial solutions of $(*)$. Prove this.

(c) Suppose $L(E, s) = \sum c_n/n^s$. Since E is modular (why?), work of Buhler, Gross, et al. gives the formula:

$$L(E, 1) = \sum c_n(exp(-2\pi nx/\sqrt{N}) + \epsilon\ exp(-2\pi n/(x\sqrt{N})))/n$$

where x is any positive real number, N is the conductor of E, and $\epsilon = \pm 1$ its root number.

Explain why this formula gives a means of computing ϵ. In the case $\epsilon = 1$, obtain a simpler formula for $L(E, 1)$.

(d) For $m = 145$, Euler claimed that $(*)$ had a nontrivial solution, namely $(159, 40)$. Show that he was mistaken.

(e) Kolyvagin proved the weak Birch Swinnerton-Dyer conjecture for modular elliptic curves over $\mathbf{Q}$ whose L-functions vanish to order at most 1 at $s = 1$. Show how this gives a way to prove that for a given m there are no nontrivial solutions. For $m = 145$ compute the coefficients of $L(E, s)$ up to $n = 10$ (the conductor of E is 48048 and root number 1) - using (c), is this enough to determine whether $(*)$ has nontrivial solutions for $m = 145$?