# MATH 844: HOMEWORK 9, DUE DEC 6.

(a) Let $E$ be an elliptic curve over $\mathbf{Q}$ with $j(E) \neq 0$. Show that $E$ has a Weierstrass equation over $\overline{\mathbf{Q}}$ of the form

$$y^2 + axy + y = x^3 \quad (a \in \overline{\mathbf{Q}})$$

and that $(0,0)$ is a torsion point on $E$. What is its order?

(b) Let $n$ be a positive integer and consider the problem of finding all nonzero integers $u, v, w$ satisfying

$$(\ast) \quad u/v + v/w + w/u = n$$

Show that this amounts to finding $E_n(\mathbf{Q})$ for a certain elliptic curve $E_n$ so long as $n \neq 3$.

(c) Show that the torsion subgroup of $E_n(\mathbf{Q})$ always has order divisible by 3. Find an $n$ for which the torsion subgroup has order larger than 3.

(d) Show that the rank of $E_6(\mathbf{Q})$ is greater than 0. Find an upper bound for this rank. Find a non-torsion point in $E_6(\mathbf{Q})$ and the corresponding solution to $(\ast)$. Show that there are infinitely many solutions to $(\ast)$ with $n = 6$ and $u, v, w > 0$.