Basic Logic

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Here are the sequent axioms and rules for BQC-2022. We have terms and predicates, including equality $x = y$. The logical symbols are $\top$, $\bot$, $A \wedge B$, $A \vee B$, $\exists x A$, and $\forall \mathbf{x}(A \to B)$. $\vec{D}$ is a finite set of formulas. Sequent axioms are rules without premise.

A1. $\vec{D}, A \Rightarrow A$ $\qquad \dfrac{\vec{D} \Rightarrow B}{\vec{D}, A \Rightarrow B}$

$$\frac{\vec{D}, A \Rightarrow B \quad \vec{D}, B \Rightarrow C}{\vec{D}, A \Rightarrow C}$$

A2. $\dfrac{\vec{D}, A, B \Rightarrow C}{\vec{D}, A \wedge B \Rightarrow C} \qquad \dfrac{\vec{D}, A \wedge B \Rightarrow C}{\vec{D}, A, B \Rightarrow C}$

A3. $\dfrac{\vec{D} \Rightarrow A \wedge B}{\vec{D} \Rightarrow A} \qquad \dfrac{\vec{D} \Rightarrow A \wedge B}{\vec{D} \Rightarrow B}$

$$\frac{\vec{D} \Rightarrow A \quad \vec{D} \Rightarrow B}{\vec{D} \Rightarrow A \wedge B}$$

A4. $\vec{D} \Rightarrow \top$

A5. $\dfrac{\vec{D}, A \vee B \Rightarrow C}{\vec{D}, A \Rightarrow C} \qquad \dfrac{\vec{D}, A \vee B \Rightarrow C}{\vec{D}, B \Rightarrow C}$

$$\frac{\vec{D}, A \Rightarrow C \quad \vec{D}, B \Rightarrow C}{\vec{D}, A \vee B \Rightarrow C}$$

A6. $\vec{D}, \bot \Rightarrow B$

A7. $\vec{D} \Rightarrow x = x$
$\vec{D}, A, x = y \Rightarrow A[x/y]$ for atoms A

A8. $\dfrac{\vec{D} \Rightarrow B}{\vec{D}[x/t] \Rightarrow B[x/t]}$ no variable of term t becomes bound

A9. $\dfrac{\vec{D}, A \Rightarrow B}{\vec{D}, \exists x A \Rightarrow B}$ x not free in $B, \vec{D}$

$$\frac{\vec{D}, \exists x A \Rightarrow B}{\vec{D}, A \Rightarrow B}$$

The fragment above with restriction to entailments $\vec{D} \Rightarrow B$ of formulas built from the atoms using only $\wedge$, $\vee$, and $\exists$, is the well-known finite geometric logic.

We write $\mathbf{x}$ for finite lists $x_1, x_2, \ldots, x_m$ of variables of length $m \geq 0$. We write $\mathbf{xy}$ for concatenated lists $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ or $\mathbf{x}y$ for $x_1, x_2, \ldots, x_m, y$. We have a universal implication $\forall \mathbf{x}(A \to B)$, where list $\mathbf{x}$ is allowed to be empty. Implication $A \to B$ is short for $\forall(A \to B)$. Negation $\neg A$ is defined by $A \to \bot$, and bi-implication $\forall \mathbf{x}(A \leftrightarrow B)$ is defined by $\forall \mathbf{x}(A \to B) \wedge \forall \mathbf{x}(B \to A)$.

A10. $\dfrac{\vec{D}, A \;\Rightarrow\; B}{\vec{D} \;\Rightarrow\; \forall \mathbf{x}(A \to B)}$ variables $\mathbf{x}$ not free in $\vec{D}$

A11. $\vec{D}, \forall \mathbf{x}(A \to B) \;\Rightarrow\; \forall \mathbf{x}y(A \to B)$ y not free left of the sequent arrow

A12. $\vec{D}, \forall \mathbf{x}y(A \to B) \;\Rightarrow\; \forall \mathbf{x}(A \to B)$

A13. $\vec{D}, \forall \mathbf{x}(A \to B), \forall \mathbf{x}(B \to C) \;\Rightarrow\; \forall \mathbf{x}(A \to C)$

A14. $\vec{D}, \forall \mathbf{x}(A \to B), \forall \mathbf{x}(A \to C) \;\Rightarrow\; \forall \mathbf{x}(A \to (B \wedge C))$

A15. $\vec{D}, \forall \mathbf{x}(B \to A), \forall \mathbf{x}(C \to A) \;\Rightarrow\; \forall \mathbf{x}((B \vee C) \to A)$

A16. $\vec{D}, \forall \mathbf{x}y(A \to B) \;\Rightarrow\; \forall \mathbf{x}(\exists y A \to B)$ y not free in B

This completes the axiomatization of BQC-2022.

We write $A \Leftrightarrow B$ as short for $A \Rightarrow B$ plus $B \Rightarrow A$. Intuitionistic Predicate Logic IQC-2022 is definable by the addition of schema $\top \to A \;\Rightarrow\; A$, which allows one to derive modus ponens. Classical Predicate Logic CQC-2022 is definable by adding $\neg\neg A \;\Rightarrow\; A$.

Proposition 0.1. *A list of derivable entailments over axioms A1 through A16.*

B1. $\vec{D} \Rightarrow B \;\dashv\vdash\; \vec{D}, \top \Rightarrow B$

B2. $A \Rightarrow B \;\vdash\; \vec{D}, A \Rightarrow B$

B3. $\vdash\; A \wedge (B \vee C) \;\Rightarrow\; (A \wedge B) \vee (A \wedge C)$

B4. $\vdash\; A \wedge \exists x B \;\Rightarrow\; \exists x(A \wedge B)$ *x not free in A*

B5. $\vdash\; A[x/s],\, s = t \;\Rightarrow\; A[x/t]$ *no variable of terms s or t becomes bound in A*

B6. $\vdash\; \forall \mathbf{x}y(A \to B) \;\Leftrightarrow\; \forall \mathbf{x}(A \to B)$ *y not free in A or B*

B7. $\vdash\; \forall \mathbf{x}y(A \to B) \;\Rightarrow\; (A[y/t] \to B[y/t])$ *no variable of term t becomes bound in A or B*

B8. $\vdash\; \forall \mathbf{x}y(A \to B) \;\Rightarrow\; \forall \mathbf{x}y(A[y/t] \to B[y/t])$ *no variable of term t becomes bound in A or B*

B9. $\vdash\; \forall x(A \to B) \;\Leftrightarrow\; (\exists x A \to B)$ *x not free in B*

0.1 Bound Variables and Formula Substitution over BQC-2022

Proposition 0.2. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two lists of variables such that they are equal as sets. Then the axiom system A1 through A16 proves* $\forall\mathbf{x}(A \to B) \Leftrightarrow \forall\mathbf{y}(A \to B)$.

Renaming bound variables is a special case of formula substitution. Let $\mathcal{L}$ be a predicate logic language, and P be a propositional letter not in $\mathcal{L}$. We write $\mathcal{L}[P]$ for the predicate logic language obtained by extending $\mathcal{L}$ with P. We write $A[P]$ for formulas over $\mathcal{L}[P]$.

Proposition 0.3 (Formula substitution)**.** *Let* $\mathcal{L}$ *be a language,* P *be a new propositional letter,* $C[P] \in \mathcal{L}[P]$*, and* $A, B \in \mathcal{L}$*. Then the axiom system A1 through A16 proves*

$$\frac{\vec{D}, A \Rightarrow B \quad \vec{D}, B \Rightarrow A}{\vec{D},\, C[A] \;\Rightarrow\; C[B]}$$

where no variable that occurs free in both $\vec{D}$ *and in* A, B *becomes bound after substitution of* A *and* B *in* $C[P]$.

Proof. We prove the claim for all $\vec{D}, A, B$ by induction on the complexity of $C[P]$. □

Proposition 0.4. *Let* C *be a formula in which the variables* x *and* y *don't occur free, and neither* x *nor* y *becomes bound after substitutions* $C[z/x]$ *or* $C[z/y]$*. Then the axiom system A1 through A16 proves* $D[\exists x C[z/x]] \Leftrightarrow D[\exists y C[z/y]]$*, for all contexts* $D[P]$.

Proposition 0.5. *Let* A *and* B *be formulas in which the variables in* $\mathbf{x}$ *and* $\mathbf{y}$ *don't occur free, and where no variable in* $\mathbf{x}$ *or* $\mathbf{y}$ *becomes bound after substitutions* $A[\mathbf{z}/\mathbf{x}]$*,* $B[\mathbf{z}/\mathbf{x}]$*,* $A[\mathbf{z}/\mathbf{y}]$*, or* $B[\mathbf{z}/\mathbf{y}]$*. Then the axiom system A1 through A16 proves* $D[\forall\mathbf{x}(A[\mathbf{z}/\mathbf{x}] \to B[\mathbf{z}/\mathbf{x}])] \Leftrightarrow D[\forall\mathbf{y}(A[\mathbf{z}/\mathbf{y}] \to B[\mathbf{z}/\mathbf{y}])]$*, for all contexts* $D[P]$.

0.2 Functional Well-formed Theories

BQC-2022 is the theory of transitive Kripke models similar to how intuitionistic predicate logic IQC-2022 is the theory of reflexive transitive Kripke models. Theories over transitive Kripke models satisfy the extra properties of being functional and well-formed.

Theories are sets of rules generated by sets BQC -2022 $\cup$ Γ, where Γ is a set of rule axioms R of form

$$R \;:=\; \frac{\vec{D}_1 \Rightarrow B_1 \;\ldots\; \vec{D}_n \Rightarrow B_n}{\vec{D}_0 \Rightarrow B_0}$$

A theory Δ is closed under derivation, equivalently $\Delta \vdash R$ exactly when $R \in \Delta$. Intersections of theories are theories. Each set of rules Γ generates a unique theory $\mathrm{Th}(\Gamma)$. A theory Δ_2 is called a *sequent theory extension* of a theory Δ_1 if there is a set of sequent axioms Γ such that Δ_2 is axiomatizable by $\Delta_1 \cup \Gamma$. So IQC-2022 and CQC-2022 axiomatize sequent theories. Let R be the rule displayed above. Then $\Gamma \vdash R$ if and only if

$$\Gamma \cup \{\vec{D}_1 \Rightarrow B_1,\ \ldots,\ \vec{D}_n \Rightarrow B_n\} \ \vdash\ \vec{D}_0 \Rightarrow B_0$$

Let $\mathcal{L} \subseteq \mathcal{M}$ be languages, and Γ be a set of rules over $\mathcal{M}$. Write $\mathcal{L} \cap \Gamma$ for the subset of rules of Γ that exist over $\mathcal{L}$.

Proposition 0.6. *Let $\mathcal{L} \subseteq \mathcal{M}$ be languages, and Δ be a theory over $\mathcal{M}$. Then $\mathcal{L} \cap \Delta$ is a theory over $\mathcal{L}$. Additionally, if $\Delta = \mathrm{Th}_{\mathcal{M}}(\Gamma)$ for some set Γ over $\mathcal{L}$, then $\mathcal{L} \cap \Delta = \mathrm{Th}_{\mathcal{L}}(\Gamma)$.*

Let R be rule

$$\frac{\vec{D}_1 \Rightarrow B_1 \ \ldots\ \vec{D}_n \Rightarrow B_n}{\vec{D}_0 \Rightarrow B_0}$$

and A be a formula. Define rule $A \times R$ by

$$A \times R \ := \ \frac{\vec{D}_1, A \Rightarrow B_1 \ \ldots\ \vec{D}_n, A \Rightarrow B_n}{\vec{D}_0, A \Rightarrow B_0}$$

Proposition 0.7. *Derivable entailments over* BQC-*2022.*

B10. $A \times (B \times R) \ \dashv\vdash\ (A \wedge B) \times R$

B11. If variables $\mathbf{z}$ *are not free in rule* R*, then* $A \times R \ \dashv\vdash\ \exists \mathbf{z} A \times R$

Let $\mathcal{L}$ be a predicate logic language, and Γ be a set of rules over $\mathcal{L}$. Set Γ is called *functional* over $\mathcal{L}$ if for all $R \in \Gamma$ and sentences $A \in \mathcal{L}$ we have $\Gamma \ \vdash\ A \times R$. All sets Γ of sequents are functional over $\mathcal{L}$.

Define $\mathcal{L} \times \Gamma = \{A \times R \mid A \in \mathcal{L} \text{ a sentence, and } R \in \Gamma\}$. By Proposition 0.7.B10 $\mathcal{L} \times \Gamma$ is functional over $\mathcal{L}$.

Proposition 0.8. *Intersections of theories functional over $\mathcal{L}$ are again functional over $\mathcal{L}$.*

Let $\Delta = \mathrm{Th}(\Gamma)$ for a set of rules Γ over $\mathcal{L}$. Then theory Δ is functional over $\mathcal{L}$ if and only if for all formulas $\vec{D}_0, B_0, \vec{D}_1, B_1, \ldots, \vec{D}_n, B_n \in \mathcal{L}$ and sentences $A \in \mathcal{L}$,

$$\Gamma \cup \{\vec{D}_1 \Rightarrow B_1, \ldots, \vec{D}_n \Rightarrow B_n\} \ \vdash\ \vec{D}_0 \Rightarrow B_0 \quad (\Gamma \vdash R)$$

implies

$$\Gamma \cup \{\vec{D}_1, A \ \Rightarrow\ B_1,\ \ldots,\ \vec{D}_n, A \ \Rightarrow\ B_n\} \ \vdash\ \vec{D}_0, A \ \Rightarrow\ B_0 \quad (\Gamma \ \vdash\ A \times R)$$

We only need this implication for rules $R \in \Gamma$:

Proposition 0.9. *A theory Δ is functional over $\mathcal{L}$ if and only if there is a functional set Γ over $\mathcal{L}$ such that $\Delta = \mathrm{Th}(\Gamma)$, that is, if and only if Δ has a functional axiomatization over $\mathcal{L}$.*

Corollary 0.10. *Theory* $\mathrm{Th}(\mathcal{L} \times \Gamma)$ *is the least theory containing* Γ *which is functional over* $\mathcal{L}$*.* BQC-*2022 is functional over* $\mathcal{L}$*. If* Δ *is a functional theory over* $\mathcal{L}$*, then so are its sequent theory extensions.*

A theory Δ over $\mathcal{L}$ is called *locally functional* over $\mathcal{L}$ if for all formulas $\vec{D}.B \in \mathcal{L}$ and sentences $A \in \mathcal{L}$ we have $\Delta \cup \{\Rightarrow A\} \vdash \vec{D} \Rightarrow B$ implies $\Delta \vdash \vec{D}, A \Rightarrow B$.

Proposition 0.11. *A theory* Δ *is functional over* $\mathcal{L}$ *if and only if all sequent theory extensions of* Δ *are locally functional over* $\mathcal{L}$*.*

Proposition 0.12. *Let* Γ *be a functional set over* $\mathcal{L}$*, and* C *be a set of new constant symbols. Let* $\mathbf{c} \in C$ *be of the same length as list of variables* $\mathbf{x}$*.*

If $\vec{D} \Rightarrow B$ *is a sequent over* $\mathcal{L}$*, then* $\Gamma \vdash \vec{D} \Rightarrow B$ *over* $\mathcal{L}$ *implies* $\Gamma \vdash \vec{D}[\mathbf{x}/\mathbf{c}] \Rightarrow B[\mathbf{x}/\mathbf{c}]$ *over* $\mathcal{L}(C)$*.*

If R *is a rule over* $\mathcal{L}$*, then* $\mathcal{L}(C) \times \Gamma \vdash R[\mathbf{x}/\mathbf{c}]$ *over* $\mathcal{L}(C)$ *implies* $\Gamma \vdash R$ *over* $\mathcal{L}$*.*

A set Γ of rules over $\mathcal{L}$ is called *well-formed* over $\mathcal{L}$ if for all rules

$$\frac{\vec{D}_1 \Rightarrow B_1 \ \ldots\ \vec{D}_n \Rightarrow B_n}{\vec{D}_0 \Rightarrow B_0} \in \Gamma$$

with all free variables among $\mathbf{x}$, and all formulas $A \in \mathcal{L}$ with no free variables among $\mathbf{x}$, we have[1]

$$\Gamma \vdash \forall\mathbf{x}(A \wedge \bigwedge \vec{D}_1 \to B_1) \wedge \ldots \wedge \forall\mathbf{x}(A \wedge \bigwedge \vec{D}_n \to B_n) \Rightarrow \forall\mathbf{x}(A \wedge \bigwedge \vec{D}_0 \to B_0)$$

Following footnote 1, we may write $\Gamma \vdash \int_{\mathbf{x}}(A \times R)$ for this entailment. Obviously all sets of sequents Γ are well-formed over $\mathcal{L}$. With Proposition 0.6 we easily verify that if $\mathcal{L} \subseteq \mathcal{M}$ are languages and Δ is a well-formed theory over $\mathcal{M}$, then $\mathcal{L} \cap \Delta$ is a well-formed theory over $\mathcal{L}$.

Proposition 0.13. *Intersections of theories well-formed over* $\mathcal{L}$ *are again well-formed over* $\mathcal{L}$*. If* Γ *is well-formed over* $\mathcal{L}$*, then so is* $\mathcal{L} \times \Gamma$*. Let* $\mathcal{M} \supseteq \mathcal{L}$ *be an extension by new function symbols (constant symbols are nullary function symbols). If* Γ *is well-formed over* $\mathcal{L}$*, then* Γ *is well-formed over* $\mathcal{M}$*.*

Proposition 0.14. *Let* R *be a rule and* $\mathbf{y}$ *be a list of variables. Then* $\int_{\mathbf{x}} R \vdash \int_{\mathbf{xy}} R$*. If none of the* $\mathbf{y}$ *are free in the (numerator) suppositions of* R*, then* $\int_{\mathbf{xy}} R \vdash \int_{\mathbf{x}} R$*.*

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[1] The sequent translation of rule R could be called $\int_{\mathbf{x}} R$, which in the definition of well-formed becomes $\int_{\mathbf{x}}(A \times R)$. A reverse derivate $(\vec{D} \Rightarrow A)'$ of 'differentiable' sequents $\vec{D} \Rightarrow A$ exists satisfying $(\int_{\mathbf{x}} R)' = R$.

Let $\Delta = \mathrm{Th}(\Gamma)$ for a set of rules Γ over $\mathcal{L}$. Then Δ is well-formed over $\mathcal{L}$ if and only if for all formulas $\vec{D}_0, B_0, \vec{D}_1, B_1, \ldots, \vec{D}_n, B_n \in \mathcal{L}$ with all free variables among $\mathbf{x}$, and all formulas $A \in \mathcal{L}$ with no free variables among $\mathbf{x}$,

$$\Gamma \cup \{\vec{D}_1 \Rightarrow B_1 \ldots \vec{D}_n \Rightarrow B_n\} \vdash \vec{D}_0 \Rightarrow B_0 \quad (\Gamma \vdash R)$$

implies

$$\Gamma \vdash \forall \mathbf{x}(A \wedge \bigwedge \vec{D}_1 \to B_1) \wedge \ldots \wedge \forall \mathbf{x}(A \wedge \bigwedge \vec{D}_n \to B_n) \Rightarrow \forall \mathbf{x}(A \wedge \bigwedge \vec{D}_0 \to B_0) \quad (\Gamma \vdash \textstyle\int_{\mathbf{x}}(A \times R))$$

We only need this implication for rules $R \in \Gamma$:

Proposition 0.15. *A theory Δ is well-formed over $\mathcal{L}$ if and only if there is a well-formed set Γ over $\mathcal{L}$ such that $\Delta = \mathrm{Th}(\Gamma)$, that is, if and only if Δ has a well-formed axiomatization over $\mathcal{L}$.*

Corollary 0.16. BQC*-2022 is well-formed over $\mathcal{L}$. If Δ is a well-formed theory over $\mathcal{L}$, then so are its sequent theory extensions.*

Proposition 0.17. *Let Γ be a well-formed set over $\mathcal{L}$, and $\mathcal{M} \supseteq \mathcal{L}$ be an extension by new function symbols (constant symbols are nullary function symbols). Then* $\mathrm{Th}(\mathcal{M} \times \Gamma)$ *is a functional well-formed theory over $\mathcal{M}$.*

1 Transitive Kripke Models for BQC-2022

A Kripke model $\mathfrak{A}$ for BQC-2022 over $\mathcal{L}$ consists of the following components. First, a structure $(W, \sqsubset)$ of a non-empty set of worlds or nodes W with transitive relation $\sqsubset$. We write $\sqsubseteq$ for the reflexive closure of $\sqsubset$. So $(W, \sqsubseteq)$ is a small category with at most one arrow between nodes. Second, a functor $k \mapsto \mathfrak{A}_k$ from $(W, \sqsubseteq)$ to the category of classical models over $\mathcal{L}$ with algebraic morphisms (preserving atoms). So for each $k \in W$ there is a classical model $\mathfrak{A}_k$ over $\mathcal{L}$, and for all pairs $k \sqsubseteq m$ there is an algebraic morphism (preserving atoms) $\restriction^k_m \colon \mathfrak{A}_k \to \mathfrak{A}_m$ such that $\restriction^k_k$ is the identity for all k, and $\restriction^m_n \restriction^k_m = \restriction^k_n$ for all $k \sqsubseteq m \sqsubseteq n$.

Given a Kripke model $\mathfrak{A}$ over $\mathcal{L}$ with node $k \in W$, classical model $\mathfrak{A}_k$ has domain A_k. We identify A_k in the usual way with a set of new constant symbols of a language $\mathcal{L}(A_k)$. We define classical truth interpretation $\mathfrak{A}_k \models B$ for sentences $B \in \mathcal{L}(A_k)$ as usual. Given $k \sqsubset m$, there is a function $\restriction^k_m \colon A_k \to A_m$, and a corresponding formula translation $B \mapsto B^k_m$ from $\mathcal{L}(A_k)$ to $\mathcal{L}(A_m)$. The formula translation is further extended to rules $R \mapsto R^k_m$ by applying the formula translation to all formulas in R

simultaneously. If $B \in \mathcal{L}(A_k)$ is an existential positive sentence, then $\mathfrak{A}_k \models B$ implies $\mathfrak{A}_m \models B^k_m$.

The inductive definition $(\mathfrak{A}, k) \Vdash B$ of forcing, for sentences $B \in \mathcal{L}(A_k)$ inductively definable by:

$(\mathfrak{A}, k) \Vdash B$ if and only if $\mathfrak{A}_k \models B$, for all atomic sentences $B \in \mathcal{L}(A_k)$
$(\mathfrak{A}, k) \Vdash B \wedge C$ if and only if $(\mathfrak{A}, k) \Vdash B$ and $(\mathfrak{A}, k) \Vdash C$
$(\mathfrak{A}, k) \Vdash B \vee C$ if and only if $(\mathfrak{A}, k) \Vdash B$ or $(\mathfrak{A}, k) \Vdash C$
$(\mathfrak{A}, k) \Vdash \exists x C$ if and only if there is $c \in A_k$ such that $(\mathfrak{A}, k) \Vdash C[x/c]$
$(\mathfrak{A}, k) \Vdash \forall \mathbf{x}(B \to C)$ if and only if for all $m \sqsupset k$ and $\mathbf{c} \in A_m$ we have $(\mathfrak{A}, m) \Vdash B^k_m[\mathbf{x}/\mathbf{c}]$ implies $(\mathfrak{A}, m) \Vdash C^k_m[\mathbf{x}/\mathbf{c}]$

So $(\mathfrak{A}, k) \Vdash \top$ and $(\mathfrak{A}, k) \nVdash \bot$, for atomic sentences $\top$ and $\bot$, and $(\mathfrak{A}, k) \Vdash B$ if and only if $\mathfrak{A}_k \models B$, for all existential positive sentences $B \in \mathcal{L}(A_k)$.

We may write $k \Vdash B$ for $(\mathfrak{A}, k) \Vdash B$ if the choice of Kripke model $\mathfrak{A}$ is clear from the context.

We extend forcing to formulas $B \in \mathcal{L}(A_k)$ with all free variables among $\mathbf{x}$ by

$k \Vdash B$ if and only if for all $m \sqsupseteq k$ and $\mathbf{c} \in A_m$ we have $m \Vdash B^k_m[\mathbf{x}/\mathbf{c}]$

For lists of formulas $\vec{D}$ with all free variables among $\mathbf{x}$ we write $k \Vdash \vec{D}$ exactly when $k \Vdash B$ for all $B \in \vec{D}$. We extend forcing to all sequents by

$k \Vdash (\vec{D} \Rightarrow B)$ if and only if for all $m \sqsupseteq k$ and $\mathbf{c} \in A_m$ we have $m \Vdash \vec{D}^k_m[\mathbf{x}/\mathbf{c}]$ implies $m \Vdash B^k_m[\mathbf{x}/\mathbf{c}]$

So $k \Vdash B$ if and only if $k \Vdash (\Rightarrow B)$.

Let R be rule

$$\frac{\vec{D}_1 \Rightarrow B_1 \ \ldots \ \vec{D}_n \Rightarrow B_n}{\vec{D}_0 \Rightarrow B_0}$$

Define

$k \Vdash R$ if and only if for all $m \sqsupseteq k$ we have $m \Vdash (\vec{D}_i \Rightarrow B_i)^k_m$ for all $i \leq n$ implies $m \Vdash (\vec{D}_0 \Rightarrow B_0)^k_m$

Finally, for sets of rules Γ we define $k \Vdash \Gamma$ if and only if $k \Vdash R$ for all $R \in \Gamma$.

For sets of rules $\Gamma \cup \{R\}$ we write $\Gamma \Vdash R$ if and only if for all transitive Kripke models $\mathfrak{A}$ and nodes k we have $(\mathfrak{A}, k) \Vdash \Gamma$ implies $(\mathfrak{A}, k) \Vdash R$.

Proposition 1.1. *Let $k \sqsubseteq m$ be nodes of a transitive Kripke model $\mathfrak{A}$, and R be a rule over $\mathcal{L}(A_k)$. Then $(\mathfrak{A}, k) \Vdash R$ implies $(\mathfrak{A}, m) \Vdash R^k_m$.*

Proposition 1.2. *Let* $\Gamma \cup \{R\}$ *be a set of rules. Then* $\Gamma \vdash R$ *implies* $\Gamma \Vdash R$.

For each node k of a transitive Kripke model $\mathfrak{A}$ we define set of rules $\mathrm{Th}(\mathfrak{A}, k)$ over $\mathcal{L}(A_k)$ by

$$\mathrm{Th}(\mathfrak{A}, k) \ := \ \{R \mid k \Vdash R\}$$

Proposition 1.3. *Let* k *be a node of transitive Kripke model* $\mathfrak{A}$*. Then* $\mathrm{Th}(\mathfrak{A}, k)$ *is a functional well-formed theory over* $\mathcal{L}(A_k)$.