Math 475.002 **Midterm Exam 2, Spring 2021**

Instructor: Mikhail Ivanov

1. (10 points) Find the number of integer solutions for

$$x + y + z = 12, \qquad 0 \le x \le 6, \quad 0 \le y \le 6, \quad 0 \le z \le 3.$$

Solution.

We seek the number of integral solutions to

$$x + y + z = 12, \qquad 0 \le x \le 6, \quad 0 \le y \le 6, \quad 0 \le z \le 3.$$

Let S denote the set of nonnegative integral solutions to $x + y + z = 12$. Let A_1 (resp. A_2, A_3) denote the set of elements in S such that $x \ge 7$ (resp. $y \ge 7$, $z \ge 4$). We seek $|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3|$. We have

set	size	justification
S	$\binom{14}{2}$	$14 = 12 + 3 - 1$
A_1	$\binom{7}{2}$	$14 - 7 = 7$
A_2	$\binom{7}{2}$	$14 - 7 = 7$
A_3	$\binom{10}{2}$	$14 - 4 = 10$
$A_1 \cap A_2$	0	$14 - 7 - 7 = 0 < 2$
$A_1 \cap A_3$	$\binom{3}{2}$	$14 - 7 - 4 = 3$
$A_2 \cap A_3$	$\binom{3}{2}$	$14 - 7 - 4 = 3$
$A_1 \cap A_2 \cap A_3$	0	$14 - 7 - 7 - 4 < 2$

By inclusion/exclusion

$$|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| = \binom{14}{2} - \binom{7}{2} - \binom{7}{2} - \binom{10}{2} + \binom{3}{2} + \binom{3}{2} = 10.$$

2. (10 points) Solve the recurrence relation $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$ $(n \ge 3)$ with initial conditions $h_0 = 0$, $h_1 = 1$, $h_2 = 2$.

Solution.

Characteristic equation: $x^3 - x^2 - 9x + 9 = 0 \Leftrightarrow (x-1)(x-3)(x+3) = 0$.
General solution: $h_4 = b \cdot 1^n + c \cdot 3^n + d(-3)^n$.
Initial conditions:

$$\begin{cases} h_0 = 0 \\ h_1 = 1 \\ h_2 = 2 \end{cases} \Leftrightarrow \begin{cases} b + c + d = 0 \\ b + 3c - 3d = 1 \\ b + 9c + 9d = 2 \end{cases} \Leftrightarrow \begin{cases} b = \frac{1}{4} \\ c = \frac{1}{3} \\ d = -\frac{1}{12}. \end{cases}$$

As a result

$$h_n = -\frac{1}{4} + \frac{1}{3}3^n - \frac{1}{12}(-3)^n.$$

3. Define sequence $\{a_n\}_{n=0}^{\infty}$ as $a_n = 3^n + 4^n$.

(a) (10 points) Find ordinary generation function for $\{a_n\}_{n=0}^{\infty}$.

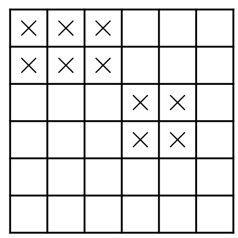
(b) (10 points) Find exponential generation function for $\{a_n\}_{n=0}^{\infty}$.

Solution.

(a) $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(3^n + 4^n)x^n = \sum_{n=0}^{\infty}(3x)^n + \sum_{n=0}^{\infty}(4x)^n = \frac{1}{1-3x} + \frac{1}{1-4x}$.

(b) $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \sum_{n=0}^{\infty}(3^n + 4^n)\frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = e^{3x} + e^{4x}$.

4. (10 points) What is the number of ways to place six non-attacking rooks on the 6×6 board with forbidden positions as shown?



Solution. We should find r_n — the number of ways to place n nonattacking rooks on the board where each of the rooks is in a forbidden position. You can do it by hand thinking how many rooks are in both components, and find $r_0 = 1$, $r_1 = 6+4 = 10$, $r_2 = 3{\cdot}2+6{\cdot}4+2{\cdot}1 = 32$, $r_3 = 3 \cdot 2 \cdot 4 + 6 \cdot 2 \cdot 1 = 36$, $r_4 = 3 \cdot 2 \cdot 2 \cdot 1 = 12$, $r5 = r6 = 0$.
Answer is

$$6! - 10 \cdot 5! + 32 \cdot 4! - 36 \cdot 3! + 12 \cdot 2!.$$

5. Number of partitions of n with at most k parts are equal to number of partitions $n + k$ with exactly k parts.
a) (5 points) Check this statement explicitly for $n = 5$, $k = 3$.
b) (10 points) Prove this statement for any positive n and k.

Solution.

(a) $n = 5$, $k = 3$ we have explicitly

$$5 \qquad 4+1 \qquad 3+2 \qquad 3+1+1 \qquad 2+2+1$$

and

$$6+1+1 \qquad 5+2+1 \qquad 4+3+1 \qquad 4+2+2 \qquad 3+3+2$$

(b) Generally, if we have a partition $n + k = a_1 + a_2 + \ldots + a_k$ in k parts, we can subtract 1 from all parts and have $n = (a_1 - 1) + (a_2 - 1) + \ldots + (a_k - 1)$ partition of n with no more than k parts (some parts can be zero).

This is bijection, because we can constact inverse map: if we have a partition $n = b_1 + b_2 + \ldots + b_\ell$ in $\ell \leq k$ parts, we can add 1 to all parts and have $n + k = (b_1 + 1) + (b_2 + 1) + \ldots + (b_\ell + 1) + 1 + 1 + \ldots + 1$ partition of $n + k$ with exactly k parts.

6. You are playing a board game in which on each turn you may advance by crawling one space, walking one space, bouncing two spaces, leaping two spaces, or flying two spaces. Let a_n be the number of different sequences of moves you can use to travel exactly n spaces.

(a) (10 points) Derive a recurrence relation and initial conditions for a_n.

(b) (10 points) Solve your recurrence relation to find a closed form for a_n. You may use any method to do so.

Solution.

(a) One can move n spaces by moving $n-2$ spaces (in any of a_{n-2} ways) and then taking either a bounce, leap, or flying; thus, there are $3a_{n-2}$ ways to travel n spaces ending with one of these three moves. Similarly, we could move $n-1$ steps in any of a_{n-1} ways, and then crawl or walk one space, giving $2a_{n-1}$ ways to walk n steps ending with either of these two moves. These exhaust the possible endings, so the total number of sequences traveling n spaces is $3a_{n-2} + 2a_{n-1}$. Thus, $a_n = 2a_{n-1} + 3a_{n-2}$. The initial conditions are easy: there is only one sequence moving zero steps (the empty sequence), so $a_0 = 1$. There are two sequences moving exactly one space: a single walk or a single crawl. Thus $a_1 = 2$.

(b) Using classical recurrence-relation solving methods, $a_n = 2a_{n-1} + 3a_{n-2}$ has characteristic polynomial $x^2 - 2x - 3$, which has roots 3 and -1. Thus, $a_n = k3^n + \ell(-1)^n$ for some k and ℓ. Plugging in the known values of a_0 and a_1, we get: $1 = k + \ell$, $2 = 3k - \ell$, so $k = \frac{3}{4}$ and $\ell = \frac{1}{4}$. Thus, $a_n = \frac{3^{n+1} + (-1)^n}{4}$.

7. Let S_n be a set of all sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_1 \le a_2 \le \ldots \le a_n$ and $a_i \le i$ for any i. For example, for $n = 3$ we have:

$$111 \quad 112 \quad 113 \quad 122 \quad 123.$$

(a) (10 points) Write down the sets S_1, S_2, S_4 and make a hypothesis about the closed form for $|S_n|$.

(b) (10 points) Prove your hypothesis from (a).

Solution.

(a)

$$S_1 = \{1\}, \qquad S_2 = \{11, 12\},$$

$$S_4 = \{1111, 1112, 1113, 1114, 1122, 1123, 1124, 1133, 1134, 1222, 1223, 1224, 1233, 1234\}.$$

So $|S_1| = 1$, $|S_2| = 2$, $|S_3| = 5$, $|S_4| = 14$, and $|S_n| = \frac{1}{n+1}\binom{2n}{n}$.

(b) There are several bijections which work well. One simple one is a bijection to the set of paths from $(1,1)$ to $(n+1,n+1)$ which are not above the diagonal. In this case a_i is simply the second coordinate of the highest point in the column i ($a_{n+1}=n+1$). So $a_i \leq i$ iff path is not above the diagonal and $a_i \leq a_{i+1}$ because we have steps only right and above.

On the other side you can prove it by recyrrence relation, where landing point is the first $i>1$ such that $a_i = i$.

8. (10 points) Let $f(n)$ denote the number of permutations of $\{1,2,\ldots,n+9,n+10\}$ such that none of the elements $\{1,2,\ldots,n\}$ occurs in its natural position. Prove that

$$(n+10)! = \sum_{k=0}^{n} \binom{n}{k} f(n-k).$$

Solution.

Let S denote the set of permutations of $\{1,2,\ldots,n+10\}$. For $0 \leq i \leq n$ let S_i denote the set of permutations in S for which exactly i integers from $\{1,2,\ldots,n\}$ are in their natural position. The sets $\{S_i\}_{i=0}^{n}$ partition S, so $|S| = \sum_{i=0}^{n} |S_i|$. We have $|S| = (n+10)!$ and $|S_i| = \binom{n}{i} f(n-i)$ for $0 \leq i \leq n$. The result follows.
Proof of $|S_i| = \binom{n}{i} f(n-i)$: To construct an elements of S_i we select the i fixed integers ($\binom{n}{i}$ ways) and select a permutation of the remaining $n+10-i$ integers without integers in natural positions (except maybe last 10 numbers). ($f(n-i)$ ways).