Math 475.002 **Midterm Exam 1, Spring 2021**
Instructor: Mikhail Ivanov

Your Name (please print) _____

Open book exam. No collaboration allowed.

The expectation is that you spend 2 hours on the exam, and then you have 20 minutes to scan and upload your work as a single PDF file. You will have 1 point penalty for every 1 minute after due time. Put your problems in the correct order (to simplify this, it might be useful to write each problem on a separate sheet of paper (certainly if you do not type your solutions)). Please also make sure all pages are in the right orientation when you convert them.

Please present your solutions in a clear manner. Justify your steps. A numerical answer without explanation cannot get credit. Cross out the writing that you do not wish to be graded on.

You do not have to carry out complicated numerical computations, but you should simplify your answer if it is possible with reasonable effort.

Problem	Value	Score
1	20	
2	30	
3	15	
4	20	
5	10	
6	10	
7	10	
Total	115	

1. Consider a word

$$\mathbb{S} = \text{COMMITTEE}$$

(a) (5 points) How many 2-permutaions of $\mathbb{S}$ are there (simplify your answer)?

(b) (5 points) How many permutations of $\mathbb{S}$ are there?

(c) (5 points) How many permutations are there if no two of letters "M" can be consecutive?

(d) (5 points) 2 letters were randomly chosen from $\mathbb{S}$. Determine the probability that there is a vowel among them.

2. Consider the following partial order on the set $X = \{2, 3, 4, 5, 6, 7, 9, 15, 30, 210\}$: $x \leq_R y$ if and only if y is divisible by x.

(a) (5 points) Draw the Hasse diagram for this poset.

(b) (5 points) How many maximal and minimal elements are there?

(c) (5 points) Find a largest chain.

(d) (5 points) Find a largest antichain.

(c) (5 points) Find a smallest chain partition.

(d) (5 points) Find a smallest antichain partition.

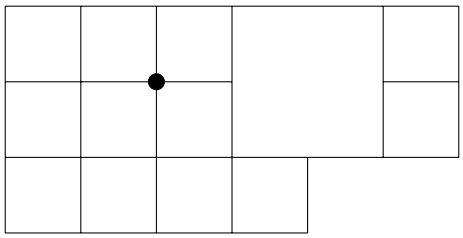
3. There are two straightforward ways to prove that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(a) (5 points) Prove it algebraically, by manipulating the two sides to get equal expressions.

(b) (10 points) Prove it combinatorially, by finding a set counted by the left side and demonstrating that the right side counts the same set.

4. Answer the following questions relating to paths through the following grid (note the excluded portions):



(a) (10 points) How many walks are there from the lower left corner to the upper right corner taking upwards and rightwards steps only?

(b) (10 points) How many of these walks pass through the point marked with a solid dot?

5. (10) Determine the number of 9-combinations of the multiset

$$\{1 \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}.$$

6.

(a) (5 points) Construct permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ whose inversion sequence is 3, 5, 1, 2, 1, 2, 0, 0.

(b) (5 points) Construct the inversion sequence of the permutation 7, 3, 5, 1, 4, 8, 2, 6.

7. (10 points) Ten students solved a total of 35 problems in a math olympiad. Each problem was solved by exactly one student. There is at least one student who solved exactly one problem, at least one student who solved exactly two problems, and at least one student who solved exactly three problems. Prove that there is also at least one student who has solved at least five problems.