Math 475.002 **Midterm Exam 1, Spring 2021**
Instructor: Mikhail Ivanov

Your Name (please print) ____

Open book exam. No collaboration allowed.

The expectation is that you spend 2 hours on the exam, and then you have 20 minutes to scan and upload your work as a single PDF file. You will have 1 point penalty for every 1 minute after due time. Put your problems in the correct order (to simplify this, it might be useful to write each problem on a separate sheet of paper (certainly if you do not type your solutions)). Please also make sure all pages are in the right orientation when you convert them.

Please present your solutions in a clear manner. Justify your steps. A numerical answer without explanation cannot get credit. Cross out the writing that you do not wish to be graded on.

You do not have to carry out complicated numerical computations, but you should simplify your answer if it is possible with reasonable effort.

Problem	Value	Score
1	20	
2	30	
3	15	
4	20	
5	10	
6	10	
7	10	
Total	115	

1. Consider a word

$$\mathbb{S} = \text{COMMITTEE}$$

(a) (5 points) How many 2-permutaions of $\mathbb{S}$ are there (simplify your answer)?

(b) (5 points) How many permutations of $\mathbb{S}$ are there?

(c) (5 points) How many permutations are there if no two of letters "M" can be consecutive?

(d) (5 points) 2 letters were randomly chosen from $\mathbb{S}$. Determine the probability that there is a vowel among them.

Solution.

(a) We have 2 types of 2-permutations. When two letters are the same, we have 3 2-permutations. When two letters are different, we have $6 \cdot 5 = 30$ permutations. Therefore, we have $30 + 3 = 33$.

(b) There are $P(2, 2, 2, 1, 1, 1) = \frac{\mathbf{9!}}{\mathbf{(2!)^3}}$ permutations.

(c) Solution 1. We have $\frac{7!}{2^2}$ permutations without M, and now 7+1=8 possible places for 2 letters M. So $\frac{7!}{2^2}\binom{8}{2}$.

Solution 2. We know number of all permutations and now we should subtrct number of permutations with consecutive M, so with "letter" "MM", i. e.

$$\frac{9!}{2!2!2!} - \frac{8!}{2!2!}.$$

(d) Denote by E our event. $\overline{E}$ is the event of obtaining two consonants. So

$$P(\overline{E}) = \frac{\binom{5}{2}}{\binom{9}{2}} \qquad \text{and} \qquad P(E) = 1 - \frac{\binom{5}{2}}{\binom{9}{2}} = \frac{13}{18}.$$

2. Consider the following partial order on the set $X = \{2, 3, 4, 5, 6, 7, 9, 15, 30, 210\}$: $x \le y$ if and only if y is divisible by x.

(a) (5 points) Draw the Hasse diagram for this poset.

(b) (5 points) How many maximal and minimal elements are there?

(c) (5 points) Find a largest chain.

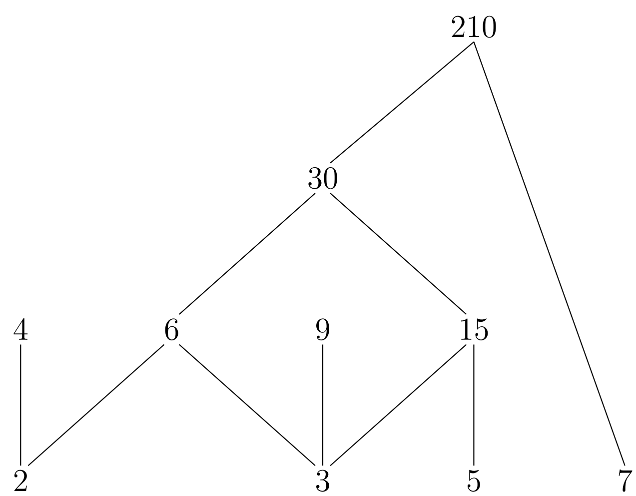
(d) (5 points) Find a largest antichain.

(e) (5 points) Find a smallest chain partition.

(f) (5 points) Find a smallest antichain partition.

Solution.

(a)



(b) The maximal elements are, 4, 9, 210. The minimal elements are 2, 3, 5, 7.

(c) For example, $\{2, 6, 30, 210\}$.

(d) For example, $\{4, 5, 6, 7, 9\}$.

(e) For example, $\{2, 4\}$, $\{3, 6, 30, 210\}$, $\{5, 15\}$, $\{7\}$, $\{9\}$.

(f) For example, $\{2, 3, 5, 7\}$, $\{4, 6, 9, 15\}$, $\{30\}$, $\{210\}$.

The Dilworth theorem states that if we found in (d) and (e) examples of the same length then this is really maximal antichain and smallest chain partition.

Dual to the Dilworth theorem states that if we found in (c) and (f) examples of the same length then this is really maximal chain and smallest antichain partition.

3. There are two straightforward ways to prove that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(a) (5 points) Prove it algebraically, by manipulating the two sides to get equal expressions.

(b) (10 points) Prove it combinatorially, by finding a set counted by the left side and demonstrating that the right side counts the same set.

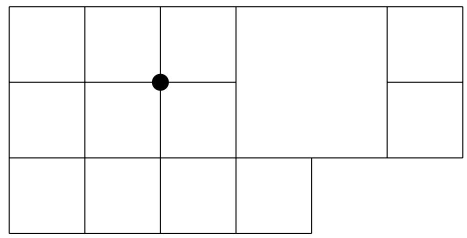
Solution.

(a)

$$\binom{n}{2} = \frac{2n(2n-1)}{2} = 2n^2 - n = n^2 + n(n-1) = n^2 + 2\binom{n}{2}.$$

(b) Consider the set $X = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$. Since $|X| = 2n$, there are $\binom{2n}{2}$ possible 2-element subsets of X. There two-element subsets could be of any of three forms: $\{a_i, a_j\}$ for distinct i and j, $\{b_i, b_j\}$ for distinct i and j, or $\{a_i, b_j\}$ for not necessarily distinct i and j. There are $\binom{n}{2}$ ways to choose two distinct elements of $\{a_1, \ldots, a_n\}$, and likewise for $\{b_1, \ldots, b_n\}$, so the first two cases account for $2\binom{n}{2}$ ways to select two-element sets. In the third case, there are n choices for the index of a_i, and n choices for the index of b_j, so there are n^2 ways to choose such a set. Thus, the number of 2-element subsets of X is $2\binom{n}{2} + n^2$.

4. Answer the following questions relating to paths through the following grid (note the excluded portions):



(a) (10 points) How many walks are there from the lower left corner to the upper right corner taking upwards and rightwards steps only?

(b) (10 points) How many of these walks pass through the point marked with a solid dot?

Solution.

(a) If the grid were complete, there would be $\binom{9}{3}$ routes from the lower left corner $(0,0)$ to the upper right $(6,3)$. We must remove those which pass through the excluded points $(4,2)$ o r $(5,0)$ (we don't need to exclude $(6,0)$ since any path through $(6,0)$ would have to go through $(5,0)$ and thus already be excluded.

Exactly $\binom{6}{2} \cdot \binom{3}{1}$ paths pass through $(4,2)$. Likewise, exactly $\binom{5}{0} \cdot \binom{4}{1}$ paths go through $(5,0)$. No paths go through both points, so no overlaping, and we get there result

$$\binom{9}{3} - \binom{6}{2}\binom{3}{1} - \binom{5}{0}\binom{4}{1} = 35.$$

(b) If the grid were complete, $\binom{4}{2}\binom{5}{1}$ paths would go through thispoint. However, we must remove those paths through this point that also go through $(4,2)$ (it is impossible to go through both the solid dot and $(5,0)$, so we don't need to worry about that exclusion. There are $\binom{4}{2}\binom{2}{0}\binom{3}{1}$ paths through both $(2,2)$ and $(4,2)$, so the number of paths through $(2,2)$ not going through $(4,2)$ is $\binom{4}{2}\binom{5}{1} - \binom{4}{2}\binom{2}{0}\binom{3}{1} = 12$, which could actually be enumerated through brute force.

5. (10) Determine the number of 9-combinations of the multiset

$$\{1 \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}.$$

Solution. For a 9-combination in question, either it contains a or it does not. The number of 9-combination containing a is equal to the numbers of 8-combinations of $\{\infty \cdot b, \infty \cdot c, \infty \cdot d\}$, which comes to $\binom{8+3-1}{3-1} = \binom{10}{2}$. The number of 9-combinations that do not contain a is equal to the number of 9-combinations of $\{\infty \cdot b, \infty \cdot c, \infty \cdot d\}$, which comes to $\binom{9+3-1}{3-1} = \binom{11}{2}$. The answer is

$$\binom{10}{2} + \binom{11}{2} = 45 + 55 = 100.$$

6.

(a) (5 points) Construct permutations of $\{1,2,3,4,5,6,7,8\}$ whose inversion sequence is 3, 5, 1, 2, 1, 2, 0, 0.

(b) (5 points) Construct the inversion sequence of the permutation 7, 3, 5, 1, 4, 8, 2, 6.

Solution.

(a) The permutation is 7, 3, 5, 1, 4, 8, 2, 6.

(b) The inversion sequence 3, 5, 1, 2, 1, 2, 0, 0.

7. (10 points) Ten students solved a total of 35 problems in a math olympiad. Each problem was solved by exactly one student. There is at least one student who solved exactly one problem, at least one student who solved exactly two problems, and at least one student who solved exactly three problems. Prove that there is also at least one student who has solved at least five problems.

Solution.

Let a_1, a_2, $\ldots$, a_{10} are numbers of problems solved by each student. WLOG $a_1 = 1$, $a_2 = 2$, $a_3 = 3$. So now we have $a_4 + a_5 + \ldots + a_{10} = 29$. If all a_i, $4 \leq i \leq 10$ are no more than 4, then $a_4 + a_5 + \ldots + a_{10} \leq 4 \cdot 7 = 28$, contadiction. So, there is i such that $a_i > 4$, i. e. $a_i \geq 5$.