HW2

11. How many sets of three integers between 1 and 20 are possible if no two consecutive integers are to be in a set?

$$\binom{18}{3}$$

$$\underbrace{\qquad\qquad\qquad}_{20}$$

$$\binom{20}{3} - 19\cdot 18 + 18$$

18 objects

$$\underbrace{\overset{x}{\frown}\, a\, \overset{y}{\frown}\, b\, \overset{z}{\frown}\, c\, \overset{t}{\frown}}$$

$$x+y+z+t=20$$

$$y, z \geq 1$$

$$\binom{18}{3} \qquad \binom{16}{3}$$

$$\underbrace{\qquad\qquad\qquad}_{20}$$

Generating permutations & inversions

Lecture 8

(Brualdi Ch. 4.1, 4.2)

Monday, September 21th

Problem: List all permutations of $\{1, 2, \ldots, n\}$

$n=2$

12
21

$n=3$

1 2 3
1 3 2
3 1 2
3 2 1
2 3 1
2 1 3

$n=4$

$$1234 \to 1243 \to 1423 \to 4123$$
$$\downarrow$$
$$1324 \leftarrow 1342 \leftarrow 1432 \leftarrow 4132$$
$$\downarrow$$
$$3124 \to 3142 \to \ldots$$

transposition

$$a_1\, a_2 \ldots a_i\, a_{i+1} \ldots a_n$$
$$\downarrow$$
$$a_1\, a_2 \ldots a_{i+1}\, a_i \ldots a_n$$

Problem: Given permutation $a_1, a_2, \dots, a_n$ what is *next* permutation in our list?

- Difficult!
- Add information

$$\overrightarrow{a_1} \; \overrightarrow{a_2} \; \cdots \; \overleftarrow{a_n}$$

$$\overrightarrow{4}\;\overleftarrow{5}\;\overleftarrow{1}\;\overleftarrow{3}\;\overleftarrow{2}$$

mobile

mobile if its arrow point to smaller number

The following algorithm generates all the permutations of $\{1, 2, 3, \ldots, n\}$ in the order that we discussed earlier.

- start with $\overleftarrow{1}\ \overleftarrow{2}\ \ldots\ \overleftarrow{n}$.

while there exists a mobile integer do:

- Find the largest mobile integer m,
- switch m with the adjacent integer it points to,
- change the direction of each integer greater then m.

Example: $n = 3$.

$$\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3} \longrightarrow \overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2} \longrightarrow \overleftarrow{3}\,\overleftarrow{1}\,\overleftarrow{2}$$

$$\downarrow$$

$$\overleftarrow{2}\,\overleftarrow{1}\,\overrightarrow{3} \longleftarrow \overleftarrow{2}\,\overrightarrow{3}\,\overleftarrow{1} \longleftarrow \overrightarrow{3}\,\overleftarrow{2}\,\overleftarrow{1}$$

$$\overrightarrow{4}\,\overleftarrow{5}\,\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}$$

$$\overleftarrow{5}\,\overrightarrow{4}\,\overleftarrow{1}\,\overleftarrow{3}\,\overleftarrow{2}$$

$$\overrightarrow{5}\,\overleftarrow{1}\,\overrightarrow{4}\,\overleftarrow{3}\,\overleftarrow{2}$$

Inversions of permutations

$n!$ permutation

$|\{0,1,\ldots,n-1\}\times\{0,1,\ldots,n-2\}\times\ldots\times\{0,1,2\}\times\{0,1\}\times\{0\}| = n!$

Definition: For a permutation $a_1, a_2, \ldots, a_n$ of $\{1, 2, 3, \ldots, n\}$ an *inversion* of this permutation is an ordered pair (a_k, a_ℓ) such that

$$k < \ell \text{ and } a_k > a_\ell.$$

45132

41, 43, 42, 51, 53, 52, 32 — 7 inv.

	perm. of 1, 2, … n
minimal	0 — 1, 2, … n
maximal	$\binom{n}{2}$ — n, n−1, …, 1

b_i = # inversions with second el. i

= # elements which $> i$ and before i

b_1	b_2	b_3	b_4	b_5
2	3	2	0	0

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~~**Example:** $n = 4$.~~

Lemma: Given a permutations $a_1, a_2, \dots, a_n$ of $\{1, 2, 3, \dots, n\}$ with inversion sequence $b_1, b_2, \dots, b_n$.

Then

$$
\begin{gathered}
0 \leqslant b_1 \leqslant n-1 \\
0 \leqslant b_2 \leqslant n-2 \\
0 \leqslant b_3 \leqslant n-3 \\
\vdots \\
0 \leqslant b_{n-1} \leqslant 1 \\
0 = b_n.
\end{gathered}
$$

Proof

$$0 \le b_i \le n - i$$

$$\underbrace{i+1, i+2, \dots, n}\ \cdot i$$

$$\left\{\begin{matrix}\text{set of} \\ \text{all} \\ \text{permutations}\end{matrix}\right\} \xrightarrow[(b_1, b_2, \dots, b_n)]{B} \left\{\begin{matrix}\{0, \dots, n-1\} \\ \{0, \dots, n-2\} \\ \vdots \\ \{0, 1, 2\} \\ \{0, 1\} \\ \{0\}\end{matrix}\right\}$$

$n!$ $\qquad$ bijection! $\qquad$ $n!$

We can recover a permutation of $\{1, 2, \ldots, n\}$ from its inversion sequence as follows.

Example: $n = 6$. Find original permutation $a_1, a_2, a_3, a_4, a_5, a_6$ from:

Method 1:

b_1	b_2	b_3	b_4	b_5	b_6
4	0	2	1	1	0

6

6 5

6 4 5

6 4 3 5

2 6 4 3 5

2 6 4 3 1 5

Method 2:

b_1	b_2	b_3	b_4	b_5	b_6
4	0	2	1	1	0

_ _ _ _ 1 _

2 _ _ _ 1 _

2 _ _ 3 1 _

2 _ 4 3 1 _

2 _ 4 3 1 5

2 6 4 3 1 5

In summary we have

Theorem: Given integer $n \geqslant 1$. The function, which sends a permutation of $\{1, 2, \ldots, n\}$ to its inversion sequence is a bijection from the set of all permutations of $\{1, 2, \ldots, n\}$ to the set
$\{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, n-2\} \times \ldots \times \{0, 1, 2\} \times \{0, 1\} \times \{0\}$.