

HW2

11. How many sets of three integers between 1 and 20 are possible if no two consecutive integers are to be in a set?

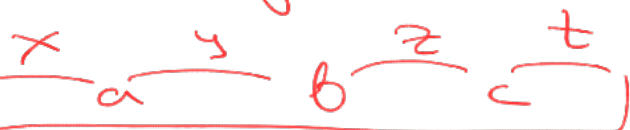
$$\binom{18}{3}$$



$$\binom{20}{3} -$$

$$- 19 \cdot 18 + 18$$

18 objects



$$x + y + z + t = 18$$

$$y, z \geq 1$$



$$\binom{18}{3} - \binom{16}{3}$$



Generating permutations & inversions

Lecture 8

(Brualdi Ch. 4.1, 4.2)

Monday, September 21th



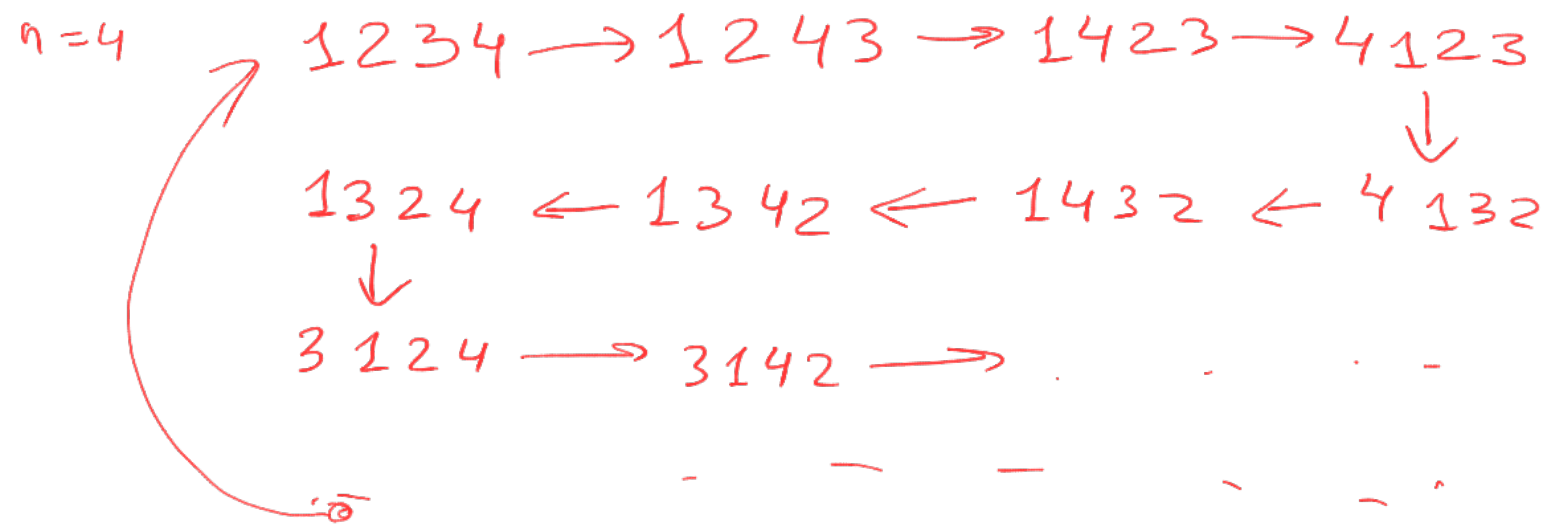
Problem: List all permutations of $\{1, 2, \dots, n\}$

$n=2$

1 2
2 1

$n=3$

1 2 3
1 3 2
3 1 2
3 2 1
2 3 1
2 1 3



transposition

$$a_1 a_2 \dots a_i a_{i+1} \dots a_n$$

$$a_1 a_2 \dots a_{i+1} a_i \dots a_n$$

Problem: Given permutation a_1, a_2, \dots, a_n what is next permutation in our list?

- Difficult!
- Add information

$a_1 \xrightarrow{\quad} a_2 \dots a_n$

$\begin{array}{ccccccc} & \rightarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \\ 4 & 5 & 1 & 3 & 2 & & \end{array}$

mobile if its arrow

point to smaller number

mobile

Inversions of permutations

$n!$ permutation

$\{0, 1, \dots, n-1\} \times \{0, 1, \dots, n-2\} \times \dots \times \{0, 1, 2\} \times \{0, 1\} \times \{0\}$
 $n!$

Definition: For a permutation a_1, a_2, \dots, a_n of $\{1, 2, 3, \dots, n\}$ an inversion of this permutation is an ordered pair (a_k, a_l) such that

$$k < l \text{ and } a_k > a_l.$$

4 5 1 3 2

41, 43, 42, 51, 53, 52, 32

Inversions

minimal	0	perm. of $1, 2, \dots, n$
maximal	$\binom{n}{2}$	$n, n-1, \dots, 1$

$b_i = \#$ inversions with second el. (c)

$= \#$ elements which $> (c)$ and before (c)

b_1	b_2	b_3	b_4	b_5
2	3	2	0	0

4 5 1 3 2

~~Example: $n=4$.~~

Lemma: Given a permutations a_1, a_2, \dots, a_n of $\{1, 2, 3, \dots, n\}$ with inversion sequence b_1, b_2, \dots, b_n .

Then

$$0 \leq b_1 \leq n-1$$

$$0 \leq b_2 \leq n-2$$

$$0 \leq b_3 \leq n-3$$

\vdots

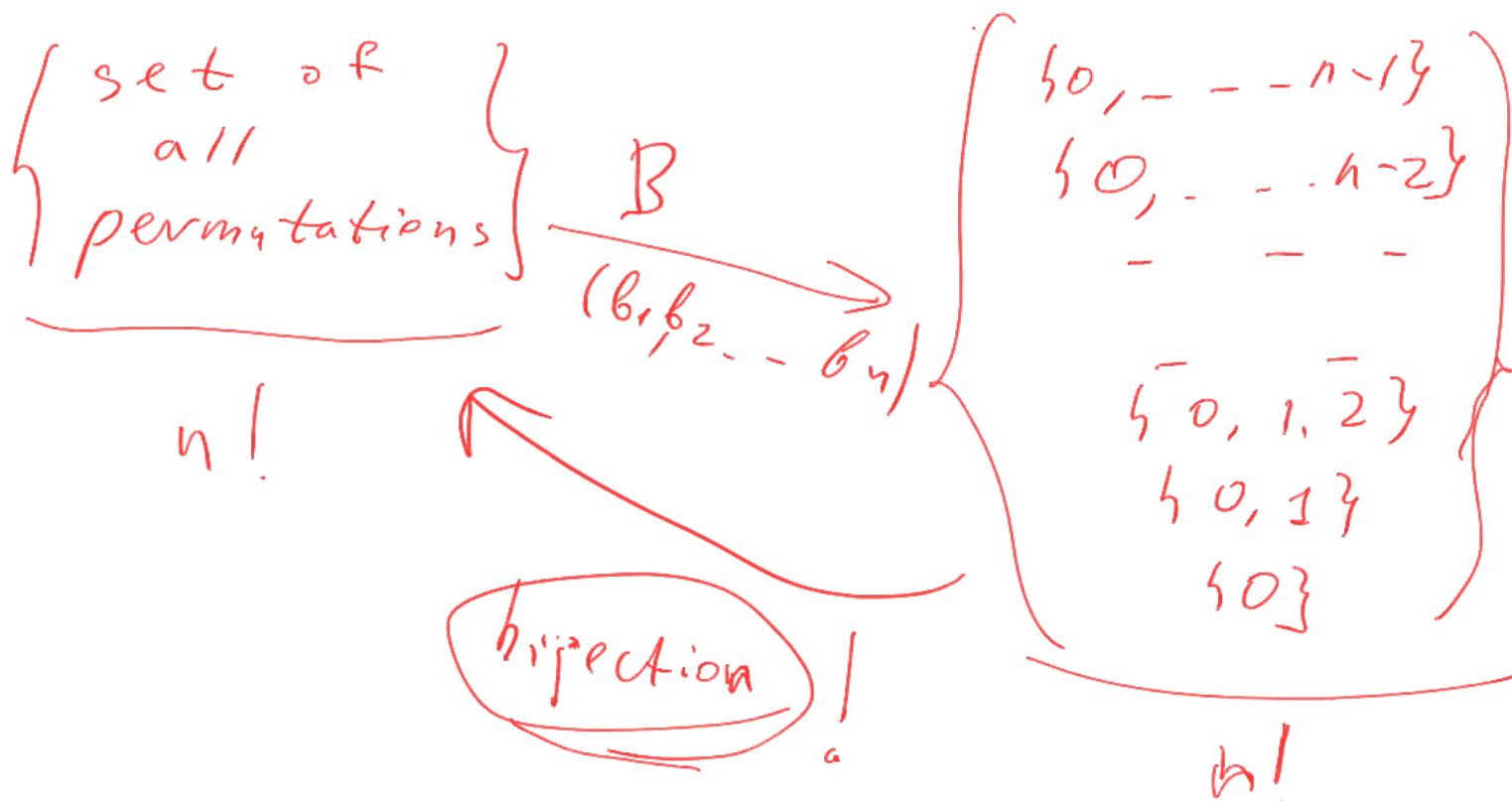
$$0 \leq b_{n-1} \leq 1$$

$$0 = b_n.$$

Proof.

$$0 \leq b_i \leq n-i$$

$i+1, i+2, \dots, n$



We can recover a permutation of $\{1, 2, \dots, n\}$ from its inversion sequence as follows.

Example: $n = 6$. Find original permutation $a_1, a_2, a_3, a_4, a_5, a_6$ from:

Method 1:

b_1	b_2	b_3	b_4	b_5	b_6
4	0	2	1	1	0

Handwritten reconstruction steps for Method 1:

6
 \wedge 6,5
 \wedge 6,4,5
 \wedge 6,4,3,5
 \wedge 2,6,4,3,5
 \wedge 2,6,4,3,1,5

Method 2:

b_1	b_2	b_3	b_4	b_5	b_6
4	0	2	1	1	0

In summary we have

Theorem: Given integer $n \geq 1$. The function, which sends a permutation of $\{1, 2, \dots, n\}$ to its inversion sequence is a bijection from the set of all permutations of $\{1, 2, \dots, n\}$ to the set

$$\{0, 1, \dots, n-1\} \times \{0, 1, \dots, n-2\} \times \dots \times \{0, 1, 2\} \times \{0, 1\} \times \{0\}.$$