HW1

16. Prove that

$$\binom{n}{r} = \binom{n}{n-r}$$

by using a combinatorial argument and not the values of these numbers as given in Theorem 2.3.1.

$$\left|\left\{ \begin{matrix} r\text{-subsets} \\ \{1,2,\ldots n\} \end{matrix} \right\}\right| = \left|\left\{ \begin{matrix} (n-r)\text{-subsets} \\ \{1,2,\ldots n\} \end{matrix} \right\}\right|$$

$$X \longrightarrow \overline{X} = \{1,2,\ldots n\} - X$$

$$\{1,2,\ldots n\} - Y \longleftarrow Y$$

26. A group of mn people are to be arranged into m teams each with n players.

(a) Determine the number of ways if each team has a different name.

(b) Determine the number of ways if the teams don't have names.

a) $\binom{mn}{n}\cdot\binom{mn-n}{n}\cdots\binom{2n}{n}\binom{n}{n} =$

$= \dfrac{(mn)!}{(n!)^m}$

4 $\{1,2,3,4\}$.

$2\cdot 2$

b) $\dfrac{(mn)!}{(n!)^m\cdot m!}$

Pigeonhole principle: first examples

Lecture 5

(Brualdi Ch. 3.1, 3.2)

Monday, September 14th

Theorem (Pigeonhole principle): For an integer $n \geqslant 1$, if we put $n+1$ objects in n boxes, then some box gets at least 2 objects.

Example: Among 13 people there are 2 that have birthdays in the same month.

$\exists$: months

Example: Assume a drawer contains a mixture of black socks and blue socks, each of which can be worn on either foot, and that you are pulling a number of socks from the drawer without looking. What is the minimum number of pulled socks required to guarantee a pair of the same color?

counter example $\to$ 2   3 $\Leftarrow$ PP   black   blue

Example: Assume a drawer contains a mixture of 10 pairs of gloves, and that you are pulling a number of gloves from the drawer without looking. What is the minimum number of pulled gloves required to guarantee a pair of gloves?

10

10 left gloves

≥ 2

11 $\Rightarrow$ ≥ 2 gloves in one pair.

Lemma: Given integer

$$a_1, a_2, \ldots, a_m \qquad (m \geqslant 1)$$

at least one of the sums

$$a_{i+1} + a_{i+2} + \ldots a_{j-1} + a_j \qquad 0 \leqslant i < j \leqslant m$$

is divisible by m.

$m = 5$ 2 4 3 1 3

$2, 4, 3, 1, 3, 2+4, 4+3, 3+1, 1+3,$

$2+4+3, 4+3+1, 3+1+3,$

$\underbrace{2+4+3+1}_{10}, 4+3+1+3, 2+4+3+1+3$

$\binom{m}{2}$ sums, $= \frac{m(m-1)}{2}$

$m-1$ numbers $\quad \overbrace{1, 1, 1, \ldots 1}^{m-1}$

$x \equiv y \pmod m \iff m \mid (x-y)$

"x equal to y modulo m"

$\{0, 1, 2, \ldots m-1\}$ $\quad a_{i+1} + \ldots + a_j \equiv 0 \pmod m$

$\exists k^{!}$ such that $\quad x \equiv k \pmod m$

Proof: $0, a_1, a_1+a_2, a_1+a_2+a_3 \ldots$

$\ldots a_1+a_2+\ldots+a_m$ $\quad m+1$ numbers

$\cancel{a_1+a_2+}\ldots+a_j \equiv \cancel{a_1+a_2+\ldots+a_i} \pmod m \quad j>i$

Definition: For a sequence $a_1, a_2, \ldots, a_n$ by a subsequence of length k we mean $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ where $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$.

This subsequence is decreasing if $a_{i_1} > a_{i_2} > \ldots > a_{i_k}$

This subsequence is increasing if $a_{i_1} < a_{i_2} < \ldots < a_{i_k}$

$n=5$ 2 5 4 1 3

	decreasing subsequences
1	2, 5, 4, 1, 3
2	21, 54, 51, 53, 41, 43
3	541, 543

Theorem (Erdös–Szekeres, 1935): Given integer $k \geqslant 0$. Let $n = k^2 + 1$, then each permutation of $\{1, 2, \ldots, n\}$ has either

at least one increasing subsequence of length $k + 1$

or

at least one decreasing subsequence of length $k + 1$.

Proof: $a_1, a_2, - - - a_n$

l_i = max length of $\nearrow$ subs. that ends at a_i

EX: $n=10$ $\quad 3,10,6,2,5,9,1,8,4,7$

a_i	3	10	6	2	5	9	1	8	4	7
l_i	1	2	2	1	2	3	1	3	2	3

We assume $\nexists \nearrow$ $k+1$

We need to show $\exists \searrow$ $k+1$

$S_l = \{a_i \mid l_i = l, \ 1 \le i \le n\}$.

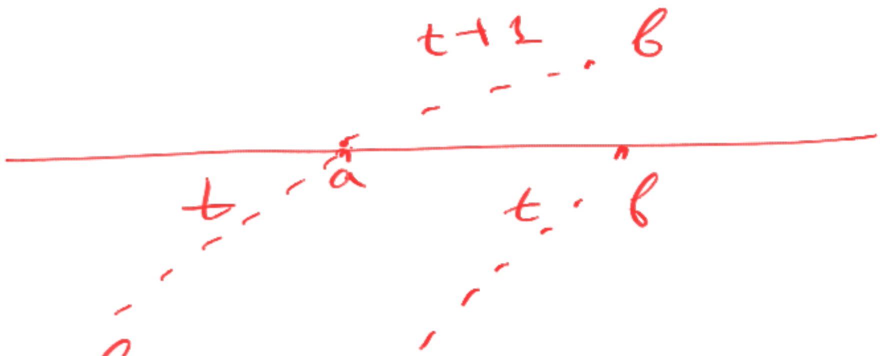
$S_1 = \{3, 2, 1\}$ $\searrow$

$S_2 = \{10, 6, 5, 4\}$ $\searrow$

$S_3 = \{9, 8, 7\}$ $\searrow$

$S_t = \{ \ldots a, b \ldots \}$ $a > b$.

If $a < b$



$\Rightarrow a > b$.

$S_1, S_2, \ldots \underline{S_k}$ — partition $\{1, 2, 3, \ldots, n\}$

$$\sum_{\ell=1}^{k} |S_\ell| = n = k^2 + 1$$

$\Rightarrow \exists \ell : |S_\ell| \geq k + 1$

$S_\ell \searrow$ subsequence.

$n = k\ell + 1$

$\Downarrow$

or

$\nearrow$ $k + 1$

$\searrow$ $\ell + 1$