Partial orders and equivalence relations

Lecture 10

(Brualdi Ch. 4.5)

Friday, September 25th

Relations

Definition: Consider a set X. Consider the Cartesian product

$$X \times X = \{(a, b) | a, b \in X\}$$

A subset

$$R \subset X \times X$$

is called a relation on X. For $a, b \in X$ write

aRb whenever $(a, b) \in R$ "a is related to b"

$a\not{R}b$ whenever $(a, b) \notin R$ "a is not related to b"

Example: Some relations

1. $X = \mathbb{Z}$, $\forall a, b \in X$ $aRb \Leftrightarrow 2|a - b$.

   $a \equiv_2 b$

2. $X =$ all subsets of $\{1, 2, \ldots, n\}$, $\forall a, b \in X$ $aRb \Leftrightarrow a \subseteq b$.

3. $X = \mathbb{Z}$, $\forall a, b \in X$ $aRb \Leftrightarrow a < b$.

We now list some important conditions on relations. Given a relation R on a set X is

Condition	Meaning ($\forall x, y, \in X$)	examples
Reflexive	xRx	1, 2
Irreflexive	$x\not{R}x$	3
Symmetric	$xRy \Rightarrow yRx$	1,
Antisymmetric	xRy and $yRx \Rightarrow x = y$	2, 3
Transitive	xRy and $yRz \Rightarrow xRz$	1, 2, 3

$a \le b$
$b \le a$
$\Downarrow$
$a = b$

Definition: Given a relation R on a set X.

$a \le b$

$a \le b.$

- Call R a *partial order* whenever R is reflexive, antisymmetric, transitive.
- The set X together with a partial order on X is called a *partially ordered set* (or *poset*).
- For $x, y \in X$, call x, y *comparable* whenever xRy or yRx.
- Call R a *total order* whenever R is a partial order and x, y are comparable for all $x, y \in X$.

1) ref
2) unsym.
3) trans
4) comparabality

Lemma: Given integer $n \geqslant 1$, $X = \{1, 2, \ldots, n\}$ and permutation a_1, a_2, $\ldots$, a_n of X. Define R by

$$xRy \Leftrightarrow x = y \text{ or } x \text{ comes before } y \text{ among } a_1, a_2, \ldots, a_n$$

Then R is a total order.

R - partial order

Definition: Call the elenent $x \in X$ *minimal* if $\nexists y \in X$ such that $y \neq x$, yRx.

Given integer $n \geqslant 1$, $X = \{1, 2, \ldots, n\}$, and total order R on X.

Observe X has unique minimal element a_1

$X \setminus \{a_1\}$ has unique minimal element a_2

$X \setminus \{a_1, a_2\}$ has unique minimal element a_1

$$\vdots$$

Then a_1, a_2, $\ldots$, a_n is a permutation of X.

Example. X,

$R = \{(x,x) \mid x \in X\}$

Now we have

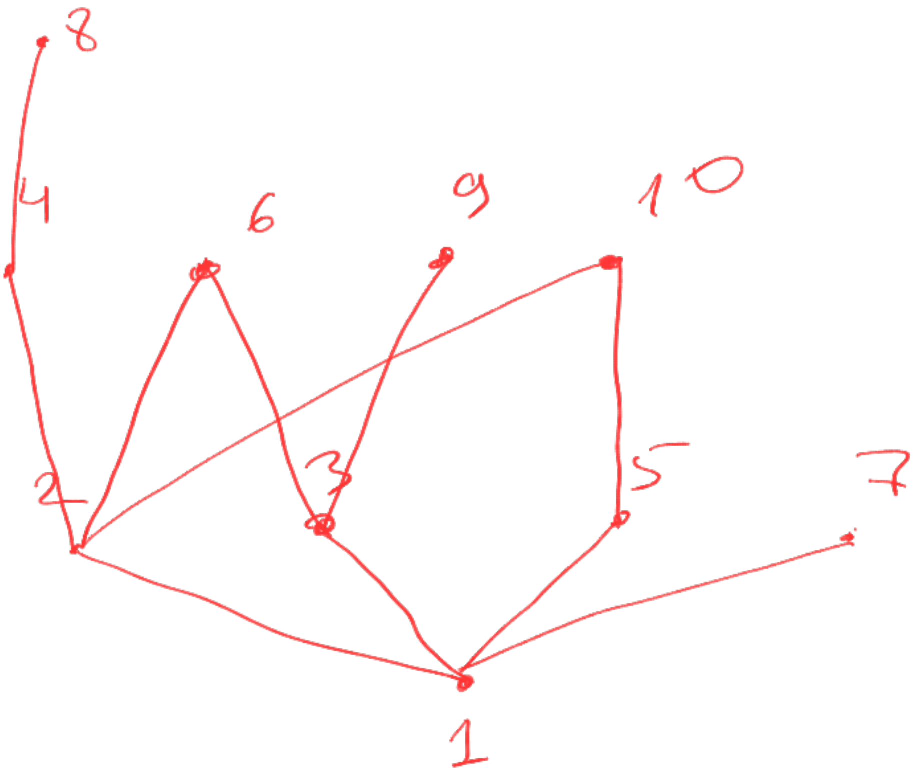
Theorem: Given integer $n \geqslant 1$, $X = \{1, 2, \ldots, n\}$. There is a one-to-one correspondence between the total orders on X and

permutations of X.

A generic partial order on a set X is usually denoted $\leqslant$.

For $x, y \in X$ write $x < y \Leftrightarrow x \neq y$ and $x \leqslant y$.

Example: Take $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define partial order $\leqslant$ on X by $x \leqslant y \Leftrightarrow x|y$.



$3 \leq 6$

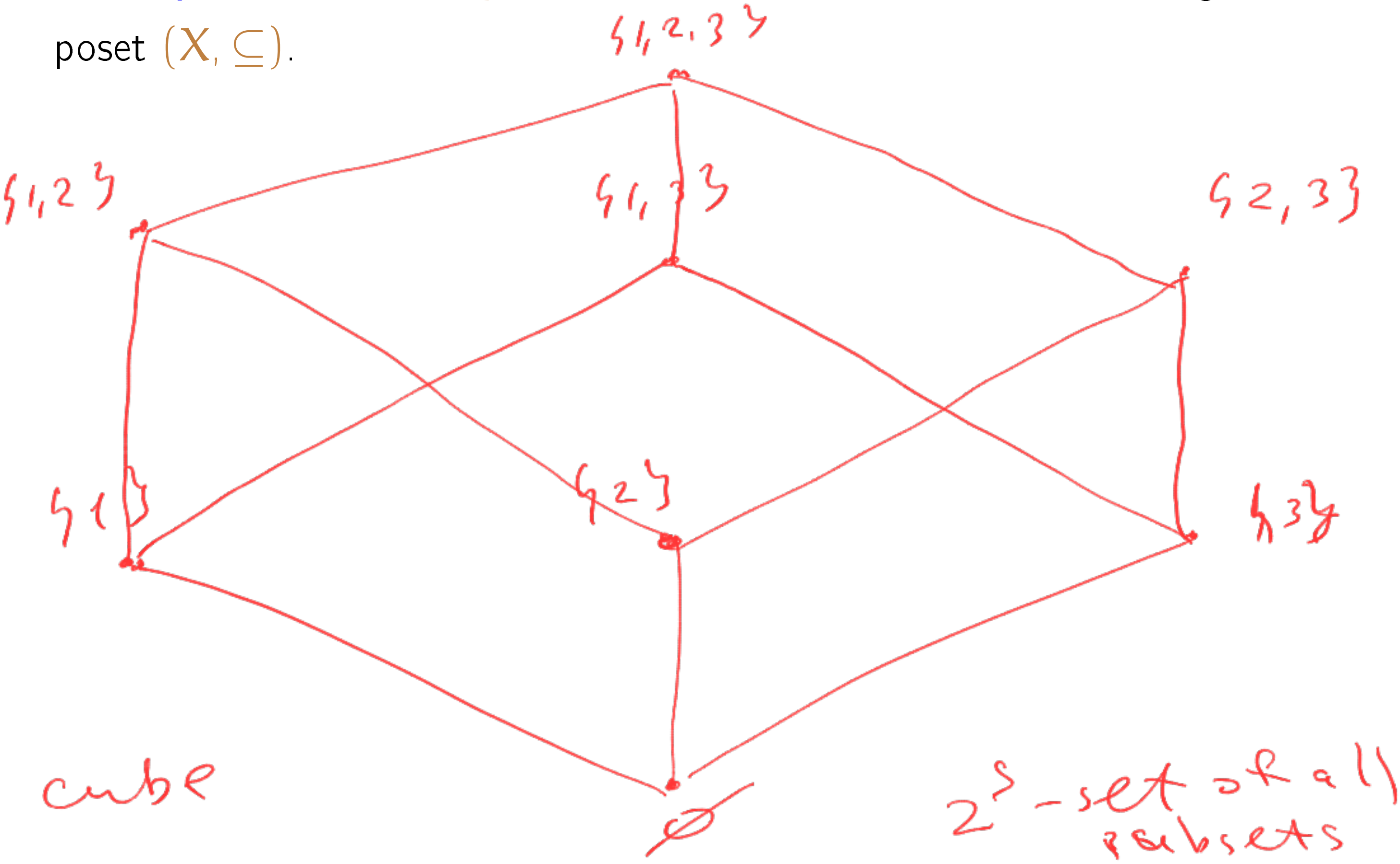
$3 \not\leq 5$

3 5 uncomp.

$X \setminus \{1\}$

Hasse diagram

Example: Take $S = \{1, 2, 3\}$. $X = 2^S$. Describe the Hasse diagram for the poset $(X, \subseteq)$.



2^S - set of all subsets

Example: Describe the Hasse diagram for the finite totally ordered sets.

Definition: Given set X. Given 2 partial orders on X:

$$\leqslant_1 \qquad \leqslant_2 .$$

$\leqslant_2$ is called an extension of $\leqslant_1$ whenever $x \leqslant_1 y$ implies $x \leqslant_2 y$ for all $x, y \in X$.

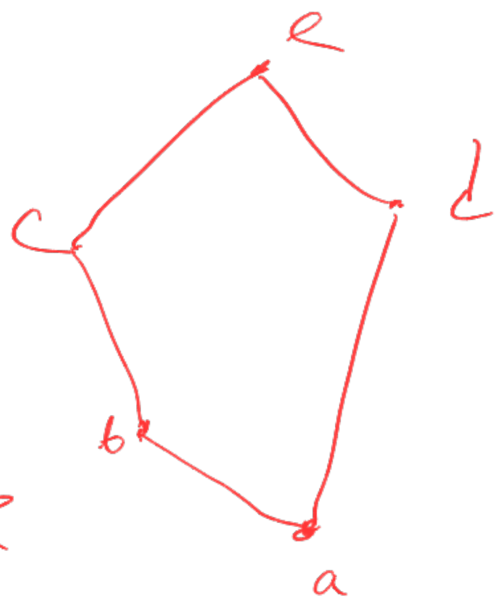
Definition: Given a partial order $\leqslant$ on a set X. A *linear extension* of $\leqslant$ is an extension of $\leqslant$ that is a total order.

Example:

$a < b < c < d < e$

$a < b < d < c < e$

$a < d < b < c < e$



Theorem: Let $\leqslant$ denote a partial order on a finite set X. Then $\leqslant$ has at least one linear extension.

$(X, \leq)$ $\quad \exists (X, \leq_{l.e.})$

$\exists$ extension

<u>Proof:</u> $\exists$ minimal element

x_1,

$X \setminus \{x_1\} \longrightarrow \exists$ min. el. x_2

$X \setminus \{x_1, x_2\} \longrightarrow \exists$ min. el. x_3

$x_1, x_2, x_3, \ldots x_n$

$\Downarrow$

we can construct total order $\leq_t$

? Is $\leq_t$ an extension of $\leq$.

$$x \leq y \overset{?}{\Longrightarrow} x \leq_t y$$

$\Downarrow$

x was chosen before

Definition: Given set X. A relation R on X is an *equivalence relation* whenever R is reflexive, symmetric, transitive.

Example: $S = \{1, 2, \ldots, n\}$. $X = 2^S$. $A, B \in X$

$$ARB \iff |A| = |B|$$

$$\mathbb{Z},\ a, b \in \mathbb{Z} \qquad aRb \iff a \equiv_m b$$

Example (X, R)

$$\Rightarrow X = X_1 \cup X_2 \cup \ldots \cup X_k$$

$$xRy \iff \exists i : x, y \in X_i$$

X_i - equivalance class

by trans.

eq. class

$$X = \underbrace{X_1 \cup X_2 \cup \ldots \cup X_k}_{\emptyset} \text{ partition}$$

$(X, \sim_{\mathcal{P}})$:

$x_1 \sim_{\mathcal{P}} x_2 \iff \exists i : x_1, x_2 \in X_i$.

Example

1) $(2^X, |\cdot|)$

$X = \{1, 2, 3\}$.

$\{\emptyset\}$

$\{\{1\}, \{2\}, \{3\}\}$

$\{\{1,2\}, \{1,3\}, \{2,3\}\}$

$\{\{1,2,3\}\}$