Math 475.003 **Exam 2, Fall 2020**
Instructor: Mikhail Ivanov

Your Name (please print) _____

Open book exam. No collaboration allowed.

The expectation is that you spend 1 hours and 30 minutes on the exam, and then you have 30 minutes to scan and upload your work as a single PDF file. Put your problems in the correct order (to simplify this, it might be useful to write each problem on a separate sheet of paper (certainly if you do not type your solutions)). Please also make sure all pages are in the right orientation when you convert them. You will earn 2 points if you will do it all and on time.

Please present your solutions in a clear manner. Justify your steps. A numerical answer without explanation cannot get credit. Cross out the writing that you do not wish to be graded on.

You do not have to carry out complicated numerical computations, but you should simplify your answer if it is possible with reasonable effort.

Problem	Value	Score
1	10	
2	10	
3	10	
4	15	
5	10	
6	18	
7	10	
8	15	
$\star$	2	
Total	100	

1. (10 points) Determine the number of 10-combinations of the multiset

$$\{3 \cdot a, 5 \cdot b, 7 \cdot c, \infty \cdot d\}.$$

2. (10 points) Solve the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}$$

for $n \geq 2$ with initial values $a_0 = 1$, $a_1 = 0$.

3. (10 points) Solve the recurrence relation

$$b_n = 8b_{n-1} - 16b_{n-2} + 4$$

for $n \geq 2$ with initial values $b_0 = 1$, $b_1 = 0$.

4. Let S_n be a set of all integers sequences $a_1, a_2, \ldots, a_n$ such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$. For example, for $n = 3$ we have:

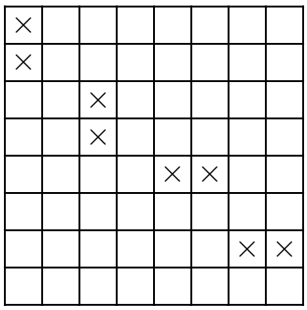
$$000 \quad 001 \quad 010 \quad 011 \quad 012.$$

(a) (5 points) Write down the sets S_1, S_2, S_4.

(b) (2 points) Find $|S_n|$.

(c) (8 points) Prove that your answer in (b) is correct.

5. (10 points) What is the number of ways to place eight non-attacking rooks on the 8×8 board with forbidden positions as shown?



6. Let W_n be a set of ways to cut $2 \times n$ board with gray and black monominoes (and) and white L-triominoes ().

(a) (6 points) Write down W_1, W_2, W_3.

(b) (6 points) Find the recurrence relation for $w_n = |W_n|$.

(c) (6 points) Find the generating function for $\{w_n\}_{n=0}^{\infty}$.

7. Six people are on a bus, and the bus will make three more stops. Everyone must get off at one of the three stops.

(a) (5 points) In how many ways this can happen?

(b) (5 points) In how many ways this can happen, if at least one person must get off at each stop?

8. For an integer $n \geq 0$ let s_n denote the number of non-negative integer solutions to

$$1^2 e_1 + 2^2 e_2 + 3^2 e_3 + \ldots + n^2 e_n = n.$$

(a) (5 points) Find s_0, s_1, s_2, ..., s_9.

(b) (5 points) Express $G(X) = \sum_{n=0}^{\infty} s_n X^n$ as a product.

(c) (5 points) Prove that your answer in (b) is correct.