Math 475.003 **Exam 1, Fall 2020**

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Your Name (please print) ____

Open book exam. No collaboration allowed

The expectation is that you spend 1 hours and 30 minutes on the exam, and then you have 30 minutes to scan and upload your work as a single PDF file.

Please present your solutions in a clear manner. Justify your steps. A numerical answer without explanation cannot get credit. Cross out the writing that you do not wish to be graded on.

You do not have to carry out complicated numerical computations, but you should simplify your answer if it is possible with reasonable effort.

Problem	Value	Score
1	25	
2	20	
3	10	
4	15	
5	10	
6	10	
7	10	
Total	100	

1. Consider a word

$$\mathbb{T} = \text{CORONAVIRUSDISEASE}$$

(a) (2 points) Is the RECURSION a 9-permutation of $\mathbb{T}$?

(b) (4 points) How many 2-permutaions of $\mathbb{T}$ are there (simplify your answer)?

(c) (4 points) How many 3-combinations of $\mathbb{T}$ are there (simplify your answer)?

(d) (3 points) How many permutations of $\mathbb{T}$ are there?

(e) (7 points) How many permutations are there if no two of letters "S" can be consecutive?

(f) (5 points) 3 letters were randomly chosen from $\mathbb{T}$. Determine the probability that all three letters are different.

Solution. $\mathbb{T}$ is

$$\{3 \cdot S, 2 \cdot O, 2 \cdot R, 2 \cdot A, 2 \cdot I, 2 \cdot E, 1 \cdot N, 1 \cdot U, 1 \cdot C, 1 \cdot D, 1 \cdot V\}$$

as a multiset.

(a) Yes, because $\{1 \cdot C, 1 \cdot O, 2 \cdot R, 1 \cdot N, 1 \cdot I, 1 \cdot U, 1 \cdot S, 1 \cdot E\} \subset \mathbb{T}$ and has 9 elements.

(b) We have 2 types of 2-permutations. If two letters are the same, we have 6 2-permutations. When two letters are different, we have $11 \cdot 10 = 110$ permutations. Therefore, we have $110 + 6 = \mathbf{116}$.

(c) We have 3 types of combinations: $\{3 \cdot x\}$, $\{2 \cdot x, 1 \cdot y\}$ and $\{1 \cdot x, 1 \cdot y, 1 \cdot z\}$. So

$$1 + \binom{6}{1}\binom{11-1}{1} + \binom{11}{3} = \mathbf{226}.$$

(d) There are

$$P(3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1) = \mathbf{\frac{18!}{3!(2!)^5}}$$

permutations.

(e) There are $P(2, 2, 2, 2, 2, 1, 1, 1, 1, 1) = \frac{15!}{(2!)^5}$ permutations of the multiset $\mathbb{T} \setminus \{3 \cdot S\}$. Than we can put "S" between any two letters, so using one of 16 positions. Answer:

$$\mathbf{\frac{15!}{(2!)^5} \cdot \binom{16}{3}}$$

Another solution. We can count complement set. If we think about "SS" as one symbol, we will have $\frac{17!}{(2!)^5}$ permutations. But we count twice permutations with "SSS" ($\frac{16!}{(2!)^5}$ permutations). So answer

$$\mathbf{\frac{18!}{3!(2!)^5} - \frac{17!}{(2!)^5} + \frac{16!}{(2!)^5}}$$

(f) Best way to work with such situations is to think that we have set

$$\{C, O_1, R_1, O_2, N, A_1, V, I_1, R_2, U, S_1, D, I_2, S_2, E_1, A_2, S_3, E_2\}$$

And find probability that we have two equal letters. it can happens if we have: $\{S_1, S_2, S_3\}$, $\{S_i, S_j, X\}$, $\{X_1, X_2, Y\}$ where X and Y are any letters. So, answer

$$1 - \frac{1 + \binom{3}{2} \cdot 15 + 5 \cdot 16}{\binom{18}{3}}.$$

2. Consider the following partial order on the set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$: $x \leq_R y$ if and only if x is a divisor of y.

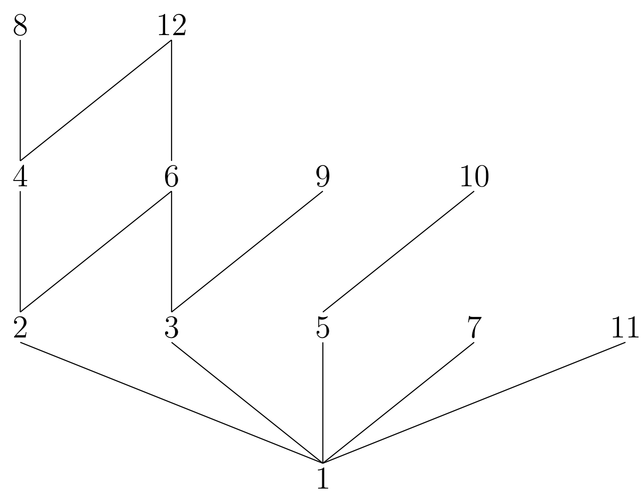
(a) (5 points) Draw the Hasse diagram for this poset.

(b) (2 points) How many maximal elements are there?

(c) (5 points) Find a largest chain (with proof).

(d) (8 points) Find a largest antichain (with proof).

Solution.

(a) 

(b) The maximal elements are 7, 8, 9, 10, 11, 12 (from the Hasse diagram).

(c) Here is a chain of size four: 1, 2, 4, 8. Here is a partition of X into four antichains: 8, 12; 4, 6, 9, 10; 2, 3, 5, 7, 11; 1. Therefore 4 is both the largest size of a chain, and the smallest number of an antichains partition.

(d) Here is an antichain of size six: 7, 8, 9, 10, 11, 12. Here is a partition of X into six chains: 1, 2, 4, 8; 3, 6, 12; 9; 5, 10; 7; 11. Therefore six is both the largest size of an antichain, and the smallest number of a chains partition.

3.

(a) (5 points) Construct permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ whose inversion sequence is 6, 6, 1, 4, 2, 1, 0, 0.

(b) (5 points) Construct the inversion sequence of the permutation 7, 3, 6, 5, 8, 4, 1, 2.

Solution.

(a) The sequence is 7,3,6,5,8,4,1,2.

(b) The inversion sequence is 6,6,1,4,2,1,0,0.

4. Prove that for any positive integers n, m, k with $n > m > k$

$$\binom{n}{k}\binom{n-k}{m-k} = \binom{m}{k}\binom{n}{m}.$$

(a) (5 points) algebraically,

(b) (10 points) using combinatorial reasoning.

Solution.

(a)

$$\binom{n}{k}\binom{n-k}{m-k} = \frac{n!}{k!(m-k)!(n-m)!} = \frac{m!}{k!(m-k)!}\frac{n!}{m!(n-m)!} = \binom{m}{k}\binom{n}{m}.$$

(b) Say we have a set S with n elements. Now, we want to choose k elements in S and another $m-k$ elements to form a new set. Therefore, we have

$$\binom{n}{k}\binom{n-k}{m-k}$$

ways to form those sets, which has m elements in total. Now, we can choose those sets another way. We first choose m elements in S, which has $\binom{n}{m}$ ways. Then we determine the k elements fror the first set, which has $\binom{m}{k}$ choices. Thus, we have

$$\binom{n}{k}\binom{n-k}{m-k} = \binom{m}{k}\binom{n}{m}.$$

Another idea.

$$LHS = \binom{n}{k}\binom{n-k}{m-k} = \binom{n}{n-k}\binom{n-k}{m-k} = P(k, m-k, n-m),$$

$$RHS = \binom{m}{k}\binom{n}{m} = \binom{n}{m}\binom{m}{m-k} = P(n-m, m-k, k).$$

5. (10 points) Provide the Gray code of order 3 starting from 010 and ending at 110.

Solution. For example:

$$010 - 011 - 001 - 000 - 100 - 101 - 111 - 110$$

(Simply we look at usual cyclic Gray code of order 3: $000 - 001 - 011 - 010 - 110 - 111 - 101 - 100$ and rewrite it starting from 010.

6. (10 points) Solve the equation for positive integer n:

$$\binom{n}{9} = \binom{n}{4}.$$

Solution. Answer: $n = 1, 2, 3, 13$. For $n = 1, 2, 3$: $\binom{n}{9} = 0 = \binom{n}{4}$. For $n = 4, 5, 6, 7, 8$: $\binom{n}{9} = 0 < \binom{n}{4}$. For $n \geq 9$ we have two elements from one row of Pascal's triangle. Row is unimodal and symmetrical, so 9 and 4 are symmetrical about $\frac{n}{2}$, i.e. $\frac{9+4}{2} = \frac{n}{2}$.

Another Solution for $n \geq 4$:

$$\frac{n!}{(n-9)!9!} = \frac{(n!}{(n-4)!4!} \Leftrightarrow (n-4)(n-5)(n-6)(n-7)(n-8) = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5.$$

LHS ia the monotonic function if $n \geq 8$, so there is no more then 1 solution. One solution is $n = 13$.

7. (10 points) There are several rocks with a total weight 10 tons. It is known that the weight of each rock is less than 1 ton. We need to deliver all these rocks to the nearest construction site. The delivery company we use has only the trucks with a weight capacity 3 tons. What is the minimum number of trucks they need to deliver all the rocks in one ride regardless of the size of the rocks?

Solution. Answer: $n = 5$. 4 is not always enough. Let's have 13 rocks, each $\frac{10}{13}$ ton. A single truck can carry at most 3 rocks ($\frac{40}{13} > 3$).
Now suppose 5 is always enough. Suppose we filled 5 trucks and have some rocks more. By Pigeonhole principle there is truck which have less than $\frac{10}{5} = 2$ tons. So we can put more rocks in this truck. Contradiction.