**Math 475.003** **Final Exam, Fall 2020**

Instructor: Mikhail Ivanov

Your Name (please print) ________________________________________

Open book exam. No collaboration allowed.

The expectation is that you spend 2 hours and 30 minutes on the exam, and then you have 30 minutes to scan and upload your work as a single PDF file. Put your problems in the correct order (to simplify this, it might be useful to write each problem on a separate sheet of paper (certainly if you do not type your solutions)). Please also make sure all pages are in the right orientation when you convert them.

Please present your solutions in a clear manner. Justify your steps. A numerical answer without explanation cannot get credit. Cross out the writing that you do not wish to be graded on. You do not have to carry out complicated numerical computations, but you should simplify your answer if it is possible with reasonable effort.

| Problem | Value | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 15 | |
| 5 | 15 | |
| 6 | 15 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 10 | |
| Total | 125 | |

**1.** Let

$$S = \text{XXXYYYZZZZ}.$$

(There are three X's, three Y 's, and four Z's).

(a) (5 points) How many permutations of the letters of $S$ are there?

(b) (5 points) How many permutations are there, if no two X's can be consecutive?

(c) (5 points) How many permutations are there, if all three of the Y's have to be consecutive?

(d) (5 points) How many permutations are there, if all three of the Y's have to be consecutive but no two X's can be consecutive?

**2.** (10 points) Determine the number of permutations of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in which no even numbers are in their natural positions.

**3.** (10 points) Solve the recurrence

$$h_n = 5h_{n-1} - 6h_{n-2} + 6n - 7, \qquad (n \geq 2)$$

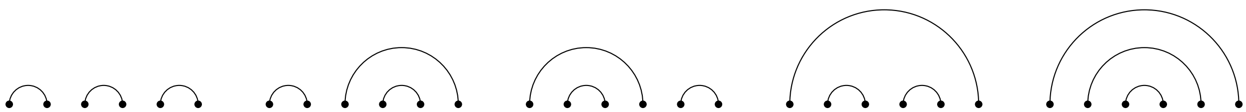
with $h_0 = 7$, $h_1 = 11$.

**4.** For an integer $n \geq 0$, let $h_n$ denote the number of $n$-digit integers with two properties (1) each digit is odd, (2) each of the digits 1, 3, 5, 7, 9 occurs an odd number of times.

(a) (5 points) Find $h_0$, $h_1$, $h_2$, $h_3$, $h_4$, $h_5$.

(b) (10 points) Find $h_n$ in a closed form.

**5.** Let $S_n$ be the set of all ways of connecting $2n$ points lying on a horizontal line by nonintersecting arcs, each arc connecting two of the points and lying above the points. For example, for $n = 3$ we have:



(a) (5 points) Write down the sets $S_1$, $S_2$, $S_4$.

(b) (2 points) Find $|S_n|$.

(c) (8 points) Prove that your answer in (b) is correct.

**6.** Let $S_n$ be the set of all lists of any length of numbers 1, 2, 3 with sum $n$ and $s_n = |S_n|$. For example, $s_2 = 2$ because $(1, 1)$ and $(2)$ are the only such lists; and $s_4 = 7$ because the lists are $(3, 1)$, $(1, 3)$, $(2, 2)$, $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$, and $(1, 1, 1, 1)$. Define $s_0 = 1$.

(a) (5 points) Determine $s_1$, $s_3$, and $s_5$ by finding all possible lists.

(b) (5 points) Give a combinatorial proof that $s_n = s_{n-1} + s_{n-2} + s_{n-3}$ for every $n \geq 3$.

(c) (5 points) Find a closed form for the generating function for $s_n$.

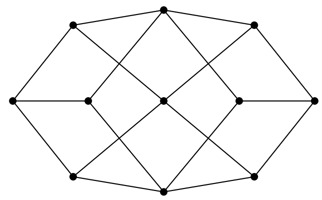
**7.** (10 points) Prove that for any nonnegative integer $n$.

$$\sum_{i=0}^{n} \binom{n}{i} \cdot \left( \binom{n+1}{i+1} + \binom{n+1}{i+2} + \ldots + \binom{n+1}{n+1} \right) = 2^{2n}$$

**8.** Let $G$ be the graph below.

(a) (5 points) Does $G$ has the Hamiltonian path?

(b) (5 points) Does $G$ has the Hamiltonian cycle?



**9.** (10 points) How many non-isomorphic trees on 6 vertices are there? Draw a picture of each tree.

**10.** (10 points) How many non-isomorphic graphs on 12 vertices with degree sequence (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)? Describe all of them.