Math 475.003 **Final Exam, Fall 2020**

Instructor: Mikhail Ivanov

Your Name (please print) ________________________________

Open book exam. No collaboration allowed.

The expectation is that you spend 2 hours and 30 minutes on the exam, and then you have 30 minutes to scan and upload your work as a single PDF file. Put your problems in the correct order (to simplify this, it might be useful to write each problem on a separate sheet of paper (certainly if you do not type your solutions)). Please also make sure all pages are in the right orientation when you convert them.

Please present your solutions in a clear manner. Justify your steps. A numerical answer without explanation cannot get credit. Cross out the writing that you do not wish to be graded on. You do not have to carry out complicated numerical computations, but you should simplify your answer if it is possible with reasonable effort.

Problem	Value	Score
1	20	
2	10	
3	10	
4	15	
5	15	
6	15	
7	10	
8	10	
9	10	
10	10	
Total	125	

1. Let

$$S = \text{XXXYYYZZZZ}.$$

(There are three X's, three Y 's, and four Z's).

(a) (5 points) How many permutations of the letters of S are there?

(b) (5 points) How many permutations are there, if no two X's can be consecutive?

(c) (5 points) How many permutations are there, if all three of the Y's have to be consecutive?

(d) (5 points) How many permutations are there, if all three of the Y's have to be consecutive but no two X's can be consecutive?

Solution.

(a) This is given by the multinomial cofficient

$$\binom{10}{3,3,4} = 4200.$$

(b) Permute the Y's and Z's $\binom{7}{3}$ ways, and then insert the X's into the 8 possible spaces including the ends $\binom{8}{3}$ ways. So in total there are

$$\binom{7}{3} \cdot \binom{8}{3} = 1960$$

ways.

(c) Glue all the Y 's together and permute X, X, X, YYY, Z, Z, Z, Z:

$$\binom{8}{3,1,4} = 280.$$

(d) Similar to the above:

$$\binom{5}{1} \cdot \binom{6}{3} = 100.$$

2. (10 points) Determine the number of permutations of $\{1,2,3,4,5,6,7,8,9\}$ in which no even numbers are in their natural positions.

Solution.

Let the set S consist of the permutations of $\{1,2,3,4,5,6,7,8,9\}$. For $i \in \{2,4,6,8\}$ let A_i denote the set of permutations in S for which i is in its natural position. We seek $|\overline{A}_2 \cap \overline{A}_4 \cap \overline{A}_6 \cap \overline{A}_8|$. We have

set	size
S	$9!$
A_i	$8!$
$A_i \cap A_j$	$7!$
$A_i \cap A_j \cap A_k$	$6!$
$A_i \cap A_j \cap A_k \cap A_\ell$	$5!$

By inclusion/exclusion

$$|\overline{A}_2 \cap \overline{A}_4 \cap \overline{A}_6 \cap \overline{A}_8| = 9! - 4 \times 8! + 6 \times 7! - 4 \times 6! + 5! = 229080.$$

3. (10 points) Solve the recurrence

$$h_n = 5h_{n-1} - 6h_{n-2} + 6n - 7, \qquad (n \geq 2)$$

with $h_0 = 7$, $h_1 = 11$.

Solution.

First find a particular solution of the nonhomogeneous equations in the form $h_n = an+b$, this gives $a = 3$ and $b = 7$, so $h_n = 3n + 7$ is a particular solution. Next, solve the homogeneous part: since the characteristic equation is $x^2 - 5x + 6 = 0$ and $x^2 - 5x + 6 = (x-2)(x-3)$, the general solution for the homogeneous part is $h_n = c \cdot 2^n + d \cdot 3^n$. Consequently, the general solution to the nonhomogeneous equation is

$$h_n = c \cdot 2^n + d \cdot 3^n + 3n + 7.$$

To find c and d, bring in initial condition and find that $c = -1$ and $d = 1$. Thus, the solution is

$$h_n = -2^n + 3^n + 3n + 7.$$

4. For an integer $n \geq 0$, let h_n denote the number of n-digit integers with two properties (1) each digit is odd, (2) each of the digits 1, 3, 5, 7, 9 occurs an odd number of times.

(a) (5 points) Find h_0, h_1, h_2, h_3, h_4, h_5.

(b) (10 points) Find h_n in a closed form.

Solution.

(a) $h_0 = h_1 = h_2 = h_3 = h_4 = 0$ and $h_5 = 120$.

(b) For $n \geq 0$, h_n is equal to the number of n-permutations of the multiset

$$\{\infty \cdot 1, \infty \cdot 3, \infty \cdot 5, \infty \cdot 7, \infty \cdot 9\}$$

such that each of 1, 3, 5, 7, 9 appears an odd number of times. The exponential generating function is

$$g_e(x) = G_1(x)G_2(x)G_3(x)G_4(x)G_5(x),$$

where

$$G_1(x) = G_2(x) = G_3(x) = G_4(x) = G_5(x) = x + \frac{x^3}{3!} + \ldots = \frac{e^x - e^{-x}}{2}.$$

Using this we obtain

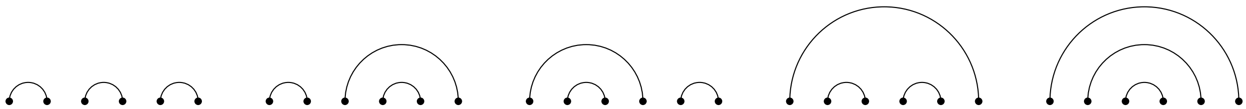
$$\begin{aligned} g_e(x) &= \frac{(e^x - e^{-x})^5}{2^5} = \frac{e^{5x} - 5e^{3x} + 10e^x - 10e^{-x} + 5e^{-3x} - e^{-5x}}{32} \\ &= \sum_{n=0}^{\infty} \frac{5^n - 5 \cdot 3^n + 10 \cdot 1^n - 10 \cdot (-1)^n + 5 \cdot (-3)^n - (-5)^n}{32} \frac{x^n}{n!}. \end{aligned}$$

Therefore

$$h_n = \frac{5^n - 5 \cdot 3^n + 10 \cdot 1^n - 10 \cdot (-1)^n + 5 \cdot (-3)^n - (-5)^n}{32}$$

for $n = 0, 1, 2, \ldots$.

5. Let S_n be the set of all ways of connecting $2n$ points lying on a horizontal line by nonintersecting arcs, each arc connecting two of the points and lying above the points. For example, for $n = 3$ we have:



(a) (5 points) Write down the sets S_1, S_2, S_4.

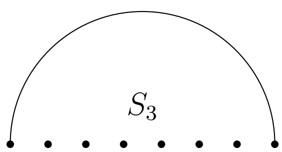
(b) (2 points) Find $|S_n|$.

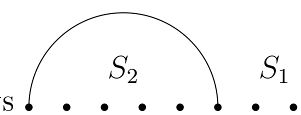
(c) (8 points) Prove that your answer in (b) is correct.

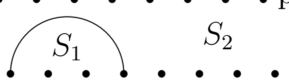
Solution.

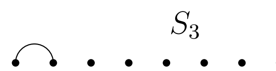
(a) S_1: 

S_2: 

S_3: 5 ways 

plus 2 ways 

plus 2 ways 

plus 5 ways 

(b) $|S_n| = C_n = \frac{\binom{2n}{n}}{n+1}$.

(c) We can partition S_n based on end of arc, starting from the first point. Then we have 2 sections, points under the arc and points outside the arc. We have a minimum of 0 points under the arc and a maximum of $2(n-1)$ points under the arc. So, if we have $2k$ points under the arc and $2(n-k-1)$ points outside, we have $|S_k| \cdot |S_{n-k-1}|$ ways to draw the rest arcs after the first one. Therefore

$$|S_n| = \sum_{k=0}^{n-1} |S_k| \cdot |S_{n-k-1}|,$$

which is the same recurrence relation as C_n. And we have the same initial value $|S_0| = C_0 = 1$.

6. Let S_n be the set of all lists of any length of numbers 1, 2, 3 with sum n and $s_n = |S_n|$. For example, $s_2 = 2$ because $(1,1)$ and (2) are the only such lists; and $s_4 = 7$ because the lists are $(3,1)$, $(1,3)$, $(2,2)$, $(2,1,1)$, $(1,2,1)$, $(1,1,2)$, and $(1,1,1,1)$. Define $s_0 = 1$.

(a) (5 points) Determine s_1, s_3, and s_5 by finding all possible lists.

(b) (5 points) Give a combinatorial proof that $s_n = s_{n-1} + s_{n-2} + s_{n-3}$ for every $n \geq 3$.

(c) (5 points) Find a closed form for the generating function for s_n.

Solution.

(a) $S_1 = \{(1)\}$, $S_3 = \{(1,1,1),(1,2),(2,1),(3)\}$, $S_5 = \{(1,1,1,1,1),(1,1,1,2),(1,1,2,1),(1,2,1,1),(2,1,1,1),(1,1,3),(1,3,1),(3,1,1),(1,2,2),(2,1,2),(2,2,1),(2,3),(3,2)\}$. So $s_1 = 1$, $s_3 = 4$, $s_5 = 13$.

(b) Consider all possible lists with the sum n. Any list my fall in one and only one of the following categories: lists ends with 1 (there are exactly s_{n-1} such lists), lists ends with 2 (there are exactly s_{n-2} such lists) and lists ends with 3 (there are exactly s_{n-3} such lists).

(c)

$$g(x) = \sum_{n=0}^{\infty} s_n x^n = s_0 + s_1 x + s_2 x^2 + \sum_{n=3}^{\infty}(s_{n-1} + s_{n-2} + s_{n-3})x^n = 1 + (x + x^2 + x^3)g(x).$$

So,

$$g(x) = \frac{1}{1 - x - x^2 - x^3}.$$

7. (10 points) Prove that for any nonnegative integer n.

$$\sum_{i=0}^{n} \binom{n}{i} \cdot \left(\binom{n+1}{i+1} + \binom{n+1}{i+2} + \ldots + \binom{n+1}{n+1} \right) = 2^{2n}.$$

Solution 1. Look at terms for i and $n-i$:

$$\begin{aligned} \binom{n}{i} \left(\binom{n+1}{i+1} + \binom{n+1}{i+2} + \ldots + \binom{n+1}{n+1} \right) + \\ \binom{n}{n-i} \left(\binom{n+1}{n-i+1} + \binom{n+1}{n-i+2} + \ldots + \binom{n+1}{n+1} \right) = \\ = \binom{n}{i} 2^{n+1}. \end{aligned}$$

So,

$$\begin{aligned} 2 \cdot \sum_{i=0}^{n} \binom{n}{i} \cdot \left(\binom{n+1}{i+1} + \binom{n+1}{i+2} + \ldots + \binom{n+1}{n+1} \right) = \\ = \sum_{i=0}^{n} \binom{n}{i} 2^{n+1} = 2^{n+1} \cdot \sum_{i=0}^{n} \binom{n}{i} = 2^{2n+1}. \end{aligned}$$

Solution 2.

Let $A := [1, n]$, $B := [n+1, 2n+1]$, then LHS counts the subsets $C \subset [1, 2n+1] =: \Omega$ with more elements in B than in A. Call this collection S, hence $C \in S \Leftrightarrow |C \cap A| < |C \cap B|$. For $C \subset \Omega$ we write $C' := \Omega \setminus C$. Then $C \in S \iff |C \cap A| < |C \cap B| \iff n - |C' \cap A| < n + 1 - |C' \cap B| \iff |C' \cap A| \geq |C' \cap B| \iff C' \notin S$. Hence for any C, exactly one element of the pair C, C' is contained in S. The result follows.

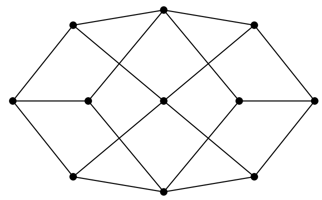
Solution 3.

It can be proved by induction.

8. Let G be the graph below.

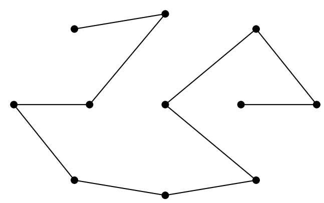
(a) (5 points) Does G has the Hamiltonian path?

(b) (5 points) Does G has the Hamiltonian cycle?

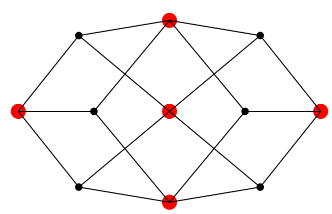


Solution.

(a) Yes. For example:



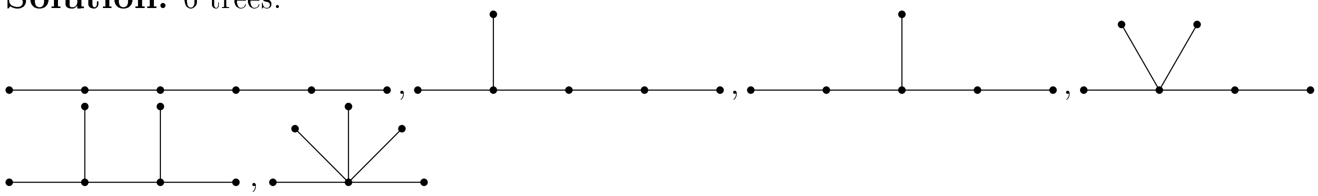
(b) Graph G is bipartite:



with sizes of parts 5 and 6. Then F does not have a Hamiltonian cycle.

9. (10 points) How many non-isomorphic trees on 6 vertices are there? Draw a picture of each tree.

Solution. 6 trees.



10. (10 points) How many non-isomorphic graphs on 12 vertices with degree sequence (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)? Describe all of them.

Solution.

Such a graph must be collection of disjoint cycles. Knowing how many cycles and of each length is enough to tell the isomorphism type. This is the same as the number of partitions of 12 into parts of size at least 3. We have

$$\begin{aligned}12 &= 12\\ &= 9+3\\ &= 8+4\\ &= 7+5\\ &= 6+6\\ &= 6+3+3\\ &= 5+4+3\\ &= 4+4+4\\ &= 3+3+3+3.\end{aligned}$$

So there are 9 isomorphism types.