

Department of Mathematics, University of Wisconsin-Madison
Math 541 — Midterm Exam II — Solutions — Fall 2025

NAME :

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Question	Points	Score
1	10	
2	8	
3	10	
4	5	
5	5	
6	10	
7	6	
8	6	
Total:	60	

INSTRUCTIONS:

Time: **90 minutes**

- This exam contains 8 questions some with multiple parts, 9 pages (including the cover) for the total of 60 points. Read the problems carefully and budget your time wisely.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others. **NO NOTES OR BOOKS** of any sort to be used too.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded.
- Unless otherwise specified, you must justify your claims and use complete sentences (or at least logical connectors indicating what you are assuming and what you are deducing).
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further. Ask for more scratch paper if needed. All scratch papers must be submitted as well.
- Please write your name on every page.

Some definitions

$$\begin{aligned}
 Z(G) &= \{g \in G \mid gx = xg \text{ for all } x \in G\} = \{g \in G \mid gxg^{-1} = x \text{ for all } x \in G\}, \\
 C_G(A) &= \{g \in G \mid ga = ag \text{ for all } a \in A\} = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}, \\
 N_G(A) &= \{g \in G \mid gA = Ag\} = \{g \in G \mid gAg^{-1} = A\}, \\
 G_s &= \{g \in G \mid g \cdot s = s\}.
 \end{aligned}$$

1. (10 points) TRUE/FALSE, no justifications needed.

(a) Given $n \in \mathbb{N}$, the alternating subgroup A_n is always a normal subgroup of S_n .

☒ **TRUE**

☐ **FALSE**

(b) Let G, H be groups. Given a homomorphism $\varphi : G \rightarrow H$, we have $H/\ker \varphi \cong G$.

☐ **TRUE**

☒ **FALSE**

(c) There is a finite number of simple groups.

☐ **TRUE**

☒ **FALSE**

(d) Let G be a group and let N be a normal subgroup of G . If both N and G/N are abelian, then G must be abelian.

☐ **TRUE**

☒ **FALSE**

(e) Let H be a subgroup of a group G . If x and y are conjugate in G , then x and y are conjugate in H .

☐ **TRUE**

☒ **FALSE**

2. (8 points) Answer Only, no justification needed. Consider the eight groups of order 4 listed below. Choose which property best describes it. You do not need to justify your answer.

(a) $\mathbb{Z}/4\mathbb{Z}$,

- ☒ Isomorphic to the cyclic group of order 4
☐ Isomorphic to the direct product of two cyclic groups of order 2.

(b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,

- ☐ Isomorphic to the cyclic group of order 4
☒ Isomorphic to the direct product of two cyclic groups of order 2.

(c) $(\mathbb{Z}/8\mathbb{Z})^\times$,

- ☐ Isomorphic to the cyclic group of order 4
☒ Isomorphic to the direct product of two cyclic groups of order 2.

(d) $(\mathbb{Z}/10\mathbb{Z})^\times$,

- ☒ Isomorphic to the cyclic group of order 4
☐ Isomorphic to the direct product of two cyclic groups of order 2.

(e) $\langle (12)(34), (13)(24) \rangle \leq S_4$,

- ☐ Isomorphic to the cyclic group of order 4
☒ Isomorphic to the direct product of two cyclic groups of order 2.

(f) $\langle i \rangle \leq \mathbb{C}^\times$,

- ☒ Isomorphic to the cyclic group of order 4
☐ Isomorphic to the direct product of two cyclic groups of order 2.

(g) $\langle i \rangle \leq Q_8$,

- ☒ Isomorphic to the cyclic group of order 4
☐ Isomorphic to the direct product of two cyclic groups of order 2.

(h) Subgroup of diagonal matrices in $GL_2(\mathbb{F}_3)$,

- ☐ Isomorphic to the cyclic group of order 4
☒ Isomorphic to the direct product of two cyclic groups of order 2.

3. (10 points) Suppose

$$\sigma_1 = (1\,2\,3\,4)(5\,6), \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 6 & 2 & 7 & 3 & 4 \end{pmatrix}.$$

(a) Is σ_1 even or odd?

Solution: even cycle is odd. Odd times odd is even.

(b) Is σ_2 even or odd?

Solution:

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 6 & 2 & 7 & 3 & 4 \end{pmatrix} = (1)(2\,5\,7\,4)(3\,6).$$

even cycle is odd. Odd times odd is even.

(c) Find one permutation $\tau \in S_7$ such that

$$\tau\sigma_1 = \sigma_2\tau.$$

Solution: We need $\tau\sigma_1\tau^{-1} = \sigma_2$. For example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 7 & 4 & 3 & 6 & 1 \end{pmatrix}$$

4. (5 points) Let G be a group, and let T be a subgroup of G with $|T| = 2$. If T is a normal subgroup of G , prove that $T \leq Z(G)$.

Solution: $T = \{e, a\}$. T is normal means $gTg^{-1} = T$:

$$gTg^{-1} = g\{e, a\}g^{-1} = \{e, gag^{-1}\} = \{e, a\},$$

so $gag^{-1} = a \Leftrightarrow ga = ag$ for every g , so $a \in Z(G)$ and $T \leq Z(G)$.

Solution: T is normal, means T is a union of conjugacy classes. Identity is own class, so non-identity element is own class too, so this element commute with any element in the group, so T is the center.

5. (5 points) Let G be a group, and let $k = |G : Z(G)|$ be the index of the center of G in G . Prove that each conjugacy class of G has at most k elements.

Solution: $Z(G) \leq C_G(g)$, so

$$k = |G : Z(G)| = |G : C_G(g)| \cdot |C_G(g) : Z(G)| \implies |G : C_G(g)| \leq k.$$

$$|\mathcal{O}_g| = |G : C_G(g)| \leq k.$$

Solution: G is a union of k left cosets of $Z(G)$: $g_1Z(G), \dots, g_kZ(G)$, $g_1 = e$. Any element in G can be written as g_iz , $z \in Z(G)$. Now consider conjugation of some element $x \in G$:

$$g_izx(g_iz)^{-1} = g_izxz^{-1}g_i^{-1} = zz^{-1}g_izg_i^{-1} = g_izg_i^{-1},$$

so there is no more than $k - 1$ possible conjugations of x ($g_1 = e$) and $|\mathcal{O}_x| \leq k$.

6. (10 points) Let D_{12} be the symmetries of the regular hexagon, with vertices labeled 1, 2, 3, 4, 5, 6. Let D_{12} act on the set $E = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$ of (unordered) pairs of opposite vertices. Let r denote rotation by $2\pi/6$ and s denote reflection about the line between vertices 1 and 4, as usual.

(a) Write down explicitly how r and s act on all three elements of E .

Solution:

$$r \cdot \{1, 4\} = \{2, 5\}, \quad r \cdot \{2, 5\} = \{3, 6\}, \quad r \cdot \{3, 6\} = \{1, 4\}.$$

$$s \cdot \{1, 4\} = \{1, 4\}, \quad s \cdot \{2, 5\} = \{3, 6\}, \quad s \cdot \{3, 6\} = \{2, 5\}.$$

(b) Is this action transitive?

Solution: Yes, r make a cycle of all elements, so all elements of E are in one orbit.

(c) Is this action faithful?

Solution: No. Element r^3 acts trivially, sending any element to itself.

(d) Explain why the permutation representation $\rho : D_{12} \rightarrow S_E \cong S_3$ is surjective. Recall that ρ is the map sending g to the function $a \mapsto g \cdot a$.)

Solution: Use part (a). Let's call diagonals 1, 2, 3. Image of r is a $(1\ 2\ 3)$. Image of s is transposition $(2\ 3)$. S_3 is generated by these 2 elements, so image is whole S_3 .

7. (6 points) Consider groups $U < GL_2(\mathbb{R})$ of upper triangular matrices and group $B < GL_2(\mathbb{R})$ of uni-upper triangular matrices

$$U = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, \quad B = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}.$$

Show that $B \trianglelefteq U$ and describe U/B in simple terms.

Solution: U consist of matrix with non-zero determinant means non-zero diagonal elements. Consider the map

$$\begin{aligned} \varphi : U &\rightarrow \mathbb{R}^\times \times \mathbb{R}^\times \\ \begin{pmatrix} a & * \\ 0 & c \end{pmatrix} &\mapsto (a, c) \end{aligned}$$

It is the homomorphism, because

$$\begin{pmatrix} a_1 & * \\ 0 & c_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & * \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & * \\ 0 & c_1 c_2 \end{pmatrix}.$$

Now $\ker(\varphi) = B$, (i.e. B is normal) $\text{im}(\varphi) = \mathbb{R}^\times \times \mathbb{R}^\times$, so, by the first theorem of isomorphism, $U/B \cong \mathbb{R}^\times \times \mathbb{R}^\times$.

Remarks:

1. It can be done by hand, showing that

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} = \begin{pmatrix} 1 & \frac{ax}{c} \\ 0 & 1 \end{pmatrix} \in B.$$

2. Actually

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}$$

means U is generated by diagonal matrices and B .

8. (6 points) Consider the group D_{12} . Let r denote rotation by $2\pi/6$ and s denote reflection about the line between vertices 1 and 4, as usual. Show that D_{12} is the internal direct product of the subgroups $\langle r^2, s \rangle$ and $\langle r^3 \rangle$.

Solution:

$$D_{12} = \{e, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}.$$

Let's denote

$$H = \langle r^2, s \rangle = \{e, r^2, r^4, s, sr^2, sr^4\}, \quad K = \langle r^3 \rangle = \{e, r^3\}.$$

It is clear that $D_{12} = HK$, $H \cap K = e$. $H \trianglelefteq D_{12}$ because $|D_{12} : H| = 2$. $K \trianglelefteq D_{12}$ because $Z(D_{12}) = K$. Now, by theorem about internal direct product group G is the direct product of subgroups H and K .

Remark: Actually, because $K \in Z(D_{12})$, we can repeat standard proof, just much easier:

$$\varphi : H \times K \rightarrow D_{12} \quad (h, k) \mapsto hk$$

is a homomorphism, because $k_1 \in Z(D_{12})$:

$$h_1k_1 \cdot h_2k_2 = h_1h_2k_1k_2.$$

First Name: _____

Last Name: _____

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM.
SCRATCH WORK WILL NOT BE GRADED