

Department of Mathematics, University of Wisconsin-Madison
Math 541 — Midterm Exam I — Solutions — Fall 2025

NAME :

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Question	Points	Score
1	10	
2	7	
3	5	
4	6	
5	5	
6	5	
7	5	
8	5	
9	6	
10	6	
Total:	60	

INSTRUCTIONS:

Time: **90 minutes**

- This exam contains 10 questions some with multiple parts, 8 pages (including the cover) for the total of 60 points. Read the problems carefully and budget your time wisely.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others. **NO NOTES OR BOOKS** of any sort to be used too.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded.
- Unless otherwise specified, you must justify your claims and use complete sentences (or at least logical connectors indicating what you are assuming and what you are deducing).
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further. Ask for more scratch paper if needed. All scratch papers must be submitted as well.
- Please write your name on every page.

Some definitions

$$\begin{aligned}
 Z(G) &= \{g \in G \mid gx = xg \text{ for all } x \in G\} = \{g \in G \mid gxg^{-1} = x \text{ for all } x \in G\}, \\
 C_G(A) &= \{g \in G \mid ga = ag \text{ for all } a \in A\} = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}, \\
 N_G(A) &= \{g \in G \mid gA = Ag\} = \{g \in G \mid gAg^{-1} = A\}, \\
 G_s &= \{g \in G \mid g \cdot s = s\}.
 \end{aligned}$$

1. (10 points) TRUE/FALSE, no justifications needed.

(a) If a permutation σ is an n -cycle, then σ^i is an n -cycle for $i = 1, \dots, n$.

- ☐ TRUE
☒ **FALSE**

(b) Let x be an element of a group with identity element 1, and assume that $x \neq 1$. If $x^{11} = 1$, then $|x| = 11$.

- ☒ **TRUE**
☐ FALSE

(c) It is possible for a group of order 8 to have exactly 5 elements of order 2.

- ☒ **TRUE**
☐ FALSE

(d) Let G be a group and $H \leq G$. Suppose that x and y are elements of G , and $x \notin H, y \notin H$. Then $xy \notin H$.

- ☐ TRUE
☒ **FALSE**

(e) Suppose that π is an element of S_7 which is equal to a four-cycle times a three-cycle, and these two cycles are disjoint. Then π^4 must be a 3-cycle.

- ☒ **TRUE**
☐ FALSE

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2. Let $\sigma = (1\ 2\ 3\ 4)(5\ 6\ 7) \in S_7$.

(a) (4 points) Compute σ^2 .

Solution:

$$\sigma^2 = (1\ 3)(2\ 4)(5\ 7\ 6).$$

(b) (3 points) What is the order of σ ? (no proof necessary)

Solution:

$$\text{lcm}(4, 3) = 12.$$

3. (5 points) Given group D_{12} the dihedral group of symmetries of a regular 6-gon. Elements r and s are in standard notation: 60° rotation and reflection by the line going through vertex 1. Simplify

$$rsr^2s^2r^3s^3r^4s^4$$

Solution: We know $s^2 = e$, $r^6 = e$, $rs = sr^{-1} = sr^5$ so

$$rsr^2s^2r^3s^3r^4s^4 = rsr^2r^3sr^4 = r \underbrace{sr^5}_{=e} sr^4 = rrsr^4 = r^6 = e.$$

4. (6 points) Let G be the set of pairs of real numbers $(a, b) \in \mathbb{R}^2$ with $a \neq 0$ and define

$$(a, b) \star (c, d) = (ac, ad + b), \quad 1_G = (1, 0).$$

Verify that this defines a group.

Solution: 1. Closure: this set is closed under this operation, $a \neq 0, c \neq 0 \Rightarrow ac \neq 0$.

2. Associativity:

$$\begin{aligned} ((a, b) \star (c, d)) \star (e, f) &= (ac, ad + b) \star (e, f) = (ace, acf + ad + b); \\ (a, b) \star ((c, d) \star (e, f)) &= (a, b) \star (ce, cf + d) = (ace, acf + ad + b). \end{aligned}$$

3. Identity:

$$\begin{aligned} (a, b) \star (1, 0) &= (a \cdot 1, a \cdot 0 + b) \\ (1, 0) \star (a, b) &= (1 \cdot a, 1 \cdot b + 0). \end{aligned}$$

4. Inverse: $(a, b)^{-1} = (\frac{1}{a}, -\frac{b}{a}), a \neq 0$.

$$\begin{aligned} (a, b) \star (\frac{1}{a}, -\frac{b}{a}) &= (a \cdot \frac{1}{a}, -a \cdot \frac{b}{a} + b) \\ (\frac{1}{a}, -\frac{b}{a}) \star (a, b) &= (\frac{1}{a} \cdot a, \frac{1}{a} \cdot b - \frac{b}{a}). \end{aligned}$$

Solution: G can be presented as a subgroup in $GL_2(\mathbb{R})$ as

$$(a, b) \mapsto \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}.$$

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5. (5 points) Are groups $\mathbb{Z}/12\mathbb{Z}$ and $(\mathbb{Z}/4\mathbb{Z})^\times$ isomorphic to each other?

Solution: No. These groups have different order:

$$|\mathbb{Z}/12\mathbb{Z}| = 12 \quad \text{and} \quad |(\mathbb{Z}/4\mathbb{Z})^\times| = \{1, 3\} = 2.$$

6. (5 points) Are groups $\mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z}/12\mathbb{Z})^\times$ isomorphic to each other?

Solution: No. $\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$, $|1| = 4$. $(\mathbb{Z}/12\mathbb{Z})^\times = \{1, 5, 7, 11\}$, $5^2 \equiv 7^2 \equiv 11^1 \equiv 1 \pmod{12}$, so on the second group there is no elements of order 4.

7. (5 points) Let G be a group, and define a function $f : G \rightarrow G$ by $f(x) = x^{-1}$. Prove that f is a homomorphism if and only if G is abelian.

Solution: Suppose G is abelian. For any $a, b \in G$

$$f(ab) = (ab)^{-1} = b^{-1}a^{-1} = f(b)f(a) = f(a)f(b),$$

so f is a homomorphism.

Now, suppose f is a homomorphism. For any $a, b \in G$

$$ab = f(a^{-1})f(b^{-1}) = f(a^{-1}b^{-1}) = f((ba)^{-1}) = ba,$$

so G is abelian.

8. (5 points) Group S_3 acts on a set $\mathbb{R}[x_1, x_2, x_3]$ of polynomials of 3 variables by

$$\begin{aligned} S_3 \times \mathbb{R}[x_1, x_2, x_3] &\longrightarrow \mathbb{R}[x_1, x_2, x_3] \\ (\sigma, P(x_1, x_2, x_3)) &\longmapsto P(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \end{aligned}$$

Find the stabilizer of a polynomial $x_1x_2 + x_3$.

Solution: We can just write down all actions:

σ	$\sigma \cdot (x_1x_2 + x_3)$
e	$x_1x_2 + x_3$
$(1\ 2)$	$x_2x_1 + x_3$
$(1\ 3)$	$x_3x_2 + x_1$
$(2\ 3)$	$x_1x_3 + x_2$
$(1\ 2\ 3)$	$x_2x_3 + x_1$
$(1\ 3\ 2)$	$x_3x_1 + x_2$

so stabilizer is $\{e, (1\ 2)\}$.

9. (6 points) Show that the center of any group G is an intersection of centralizers of all elements:

$$Z(G) = \bigcap_{g \in G} C_G(g)$$

Solution:

$$h \in C_G(g) \Rightarrow hg = gh$$

$$h \in \bigcap_{g \in G} C_G(g) \Rightarrow hg = gh \text{ for any } g \in G$$

so

$$\bigcap_{g \in G} C_G(g) \subseteq Z(G).$$

In the opposite direction:

$$Z(G) \subseteq C_G(g)$$

so

$$Z(G) \subseteq \bigcap_{g \in G} C_G(g).$$

Solution: We can show that $Z(G) = C_G(G)$ and $C_G(A) \cap C_G(B) = C_G(A \cup B)$.

10. (6 points) Let G be a group, let $H \leq G$ and $g \in G$. Prove that identity for normalizers

$$N_G(gHg^{-1}) = gN_G(H)g^{-1}.$$

Solution:

$$\begin{aligned} x \in gN_G(H)g^{-1} &\iff g^{-1}xg \in N_G(H) \iff g^{-1}xgH = Hg^{-1}xg \iff \\ & \quad xgH = gHg^{-1}xg \iff xgHg^{-1} = gHg^{-1}x \iff x \in N_G(gHg^{-1}). \end{aligned}$$

Solution: Let $x \in N_G(H)$, i.e. $xhx^{-1} \in H$ for any $h \in H$. Elements of gHg^{-1} are ghg^{-1} . We need to show that elements of the form gxg^{-1} normalize gHg^{-1} :

$$gxg^{-1} \cdot ghg^{-1} \cdot (gxg^{-1})^{-1} = gxg^{-1}ghg^{-1}gx^{-1}g^{-1} = gxhx^{-1}g^{-1} = g(xhx^{-1})g^{-1} \in gHg^{-1}.$$

Opposite side is similar.

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM.
SCRATCH WORK WILL NOT BE GRADED