Department of Mathematics, University of Wisconsin-Madison

Math 475 — Midterm Exam 2 — Fall 2023

NAME : (as it appears on Canvas)

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INSTRUCTIONS:

Time: **90 minutes**

Please write your name on every page

Read the problems carefully and budget your time wisely.

No calculators or other electronic devices, please. **Turn off your phone.**

Please present your solutions in a clear manner. Justify your steps.
A numerical answer without explanation cannot get credit.
Cross out the writing that you do not wish to be graded on.

You must use correct notation to receive full credit.

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	12	12	12	12	12	16	12	100

1. (12 points) Determine the number of 10-combinations of the multiset

$$\{5 \cdot a, 5 \cdot b, 5 \cdot c, \infty \cdot d\}.$$

2. (12 points) Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which exactly four integers are in their natural positions.

3. (12 points) What is the number of ways to place six non-attacking rooks on the 6×6 board with forbidden positions as shown?

×					
×					
		×			
		×			
				×	×

On this page, only the answer will be graded. MARK YOUR ANSWER CLEARLY. But you do not need to justify your work.

4. (12 points) Determine the exponential generating function for number of ways to color the squares of a $1 \times n$ chessboard, using the colors red, blue and green if an even number of squares is to be colored red and an even number is to be colored green.

5. (12 points) Let S be the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \infty \cdot e_3\}$. Determine the generating function for the sequence h_0, h_1, h_2, $\ldots$, h_n, $\ldots$, where h_n is the number of n-combinations of S with the following added restrictions:

   - e_1 occurs an odd number of times.
   - e_2 occurs at most two.
   - e_3 occurs 1, 3, or 6 times.

6. (12 points) Solve the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}$$

for $n \geq 2$ with initial values $a_0 = 1$, $a_1 = 0$.

7. Let W_n be a set of ways to cut $1 \times n$ board with gray and black monominoes (■ and ■) and white dominoes (□□).

   (a) (6 points) Write down W_1, W_2, W_3.

   (b) (10 points) Find the recurrence relation for $w_n = |W_n|$.

8. Six people are on a bus, and the bus will make three more stops. Everyone must get off at one of the three stops.
   (a) (4 points) In how many ways this can happen?
   (b) (8 points) In how many ways this can happen, if at least one person must get off at each stop? Numeric answer is required.