Department of Mathematics, University of Wisconsin-Madison

Math 475 — Midterm Exam 2 — Fall 2023

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INSTRUCTIONS:

Time: **90 minutes**

Please write your name on every page

Read the problems carefully and budget your time wisely.

No calculators or other electronic devices, please. **Turn off your phone.**

Please present your solutions in a clear manner. Justify your steps.
A numerical answer without explanation cannot get credit.
Cross out the writing that you do not wish to be graded on.

You must use correct notation to receive full credit.

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	12	12	12	12	12	16	12	100

1. (12 points) Determine the number of 10-combinations of the multiset

$$\{5 \cdot a, 5 \cdot b, 5 \cdot c, \infty \cdot d\}.$$

Solution: Let our combination be $\{x \cdot a, y \cdot b, z \cdot c, t \cdot d\}$. Let

$$A = \{(x, y, z, t) \mid x + y + z + w = 10, x, y, z, , t \geq 0\},$$

$$A_1 = \{(x, y, z, t) \in A | x \geq 6\},$$
$$A_2 = \{(x, y, z, t) \in A | y \geq 6\},$$
$$A_3 = \{(x, y, z, t) \in A | z \geq 6\}.$$

Answer:

$$= |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |A| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_2| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$$

$$= \binom{10+3}{3} - \left(\binom{4+3}{3} + \binom{4+3}{3} + \binom{4+3}{3}\right) + (0+0+0) - 0 = \binom{13}{3} - 4\binom{7}{3} = 181.$$

2. (12 points) Determine the number of permutations of $\{1, 2, \dots, 8\}$ in which exactly four integers are in their natural positions.

Solution: Let X denote the set of permutations of $\{1, 2, \dots, 8\}$ for which exactly four integers are in their natural position. We compute $|X|$. To do this we construct an element of X in stages:

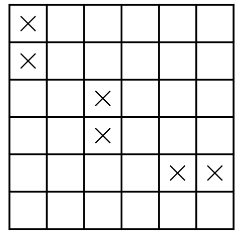
stage	to do	# choices
1	select the four fixed integers	$\binom{8}{4}$
2	select a derangement of the remaining four integers	D_4

Therefore $|X| = \binom{8}{4} D_4$. We have $\binom{8}{4} = 70$ and

$$D_4 = 4! - 4 \times 3! + 6 \times 2! - 4 \times 1! + 1 = 9,$$

so $|X| = 70 \times 9 = 630$.

3. (12 points) What is the number of ways to place six non-attacking rooks on the 6×6 board with forbidden positions as shown?



Solution: By the Rook Theorem, we should find r_n — the number of ways to place n nonattacking rooks on the board where each of the rooks is in a forbidden position.

You can do it by hand, and find $r_0 = 1$, $r_1 = 6$, $r_2 = 12$, $r_3 = 8$, $r_4 = r_5 = r_6$.

Or, you can find them, using generating functions, because we have 3 independent parts, and for one part we have $r_0 = 1$, $r_1 = 2$, $r_2 = 0$, for all diagram we have

$$\sum_{i=0}^{6} r_i x^i = (1 + 2x)^3.$$

In any way, answer is

$$6! - 6 \cdot 5! + 12 \cdot 4! - 8 \cdot 3! = 240.$$

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On this page, only the answer will be graded. MARK YOUR ANSWER CLEARLY. But you do not need to justify your work.

4. (12 points) Determine the exponential generating function for number of ways to color the squares of a $1 \times n$ chessboard, using the colors red, blue and green if an even number of squares is to be colored red and an even number is to be colored green.

Solution:

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\dots\right)\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right) = \left(\frac{e^x+e^{-x}}{2}\right)\cdot e^x \cdot \left(\frac{e^x+e^{-x}}{2}\right)$$

5. (12 points) Let S be the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \infty \cdot e_3\}$. Determine the generating function for the sequence h_0, h_1, h_2, $\dots$, h_n, $\dots$, where h_n is the number of n-combinations of S with the following added restrictions:

- e_1 occurs an odd number of times.
- e_2 occurs at most two.
- e_3 occurs 1, 3, or 6 times.

Solution:

$$(x+x^3+x^5+\dots)(1+x+x^2)(1+x^3+x^6) = \frac{x}{1-x^2}(1+x+x^2)(1+x^3+x^6)$$

6. (12 points) Solve the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}$$

for $n \geq 2$ with initial values $a_0 = 1$, $a_1 = 0$.

Solution:

Characteristic equation: $x^2 - 8x + 16 = 0 \Leftrightarrow (x-4)^2 = 0$.

General solution: $a_4 = b \cdot 4^n + c \cdot n4^n$.

Initial conditions:

$$\begin{cases} a_0 = 1 \\ a_1 = 0 \end{cases} \Leftrightarrow \begin{cases} b = 1 \\ 4b + 4c = 0 \end{cases} \Leftrightarrow \begin{cases} b = 1 \\ c = -1. \end{cases}$$

As a result

$$a_n = 4^n - n4^n.$$

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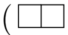
7. Let W_n be a set of ways to cut $1 \times n$ board with gray and black monominoes ( and ) and white dominoes ().

   (a) (6 points) Write down W_1, W_2, W_3.

Solution:

$W_1 = \{$  $\}$

$W_2 = \{$  $\}$

$W_3 = \{$  $\}$

   (b) (10 points) Find the recurrence relation for $w_n = |W_n|$.

Solution: Consider the set W_n and partition it by the color of the last square. If this color is gray, then this is a gray monomino and quantity of such ways is equal to the w_{n-1}. Similarly if the color is black then we have w_{n-1}. If this color is white, then this is a white domino and quantity of such ways is equal to the w_{n-2}. So

$$w_n = 2w_{n-1} + w_{n-2}.$$

8. Six people are on a bus, and the bus will make three more stops. Everyone must get off at one of the three stops.

   (a) (4 points) In how many ways this can happen?

   Solution: Answer: $3^6 = 729$. Any person can use at one of the three stops.

   (b) (8 points) In how many ways this can happen, if at least one person must get off at each stop? Numeric answer is required.

Solution:

Solution 1: We should divide our set in 3 non-empty parts with labels 1, 2 and 3. So

$$3! \cdot S(6,3) = 6 \cdot 90 = 540.$$

Solution 2: We can use inclusion-exclusion formula for 3 sets:

$$3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 = 729 - 192 + 3 = 540.$$

Solution 3: We can divide 6 in three positive parts 3 ways: 6=3+2+1=4+1+1=2+2+2. So

$$3! \cdot \frac{6!}{3!2!1!} + 3 \cdot \frac{6!}{4!1!1!} + \frac{6!}{2!2!2!} = 6 \cdot 60 + 3 \cdot 30 + 90 = 540.$$