Department of Mathematics, University of Wisconsin-Madison

Math 475 — Midterm Exam 1 — Fall 2023

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INSTRUCTIONS:

Time: **90 minutes**

Please write your name on every page

Read the problems carefully and budget your time wisely.

No calculators or other electronic devices, please. **Turn off your phone.**

Please present your solutions in a clear manner. Justify your steps.
A numerical answer without explanation cannot get credit.
Cross out the writing that you do not wish to be graded on.

You must use correct notation to receive full credit.

Question:	1	2	3	4	5	6	7	8	Total
Points:	15	20	15	10	10	10	10	10	100

1. Consider a word

$$\mathbb{S} = \text{WISCONSIN}$$

(a) (5 points) How many 2-permutaions of $\mathbb{S}$ are there (simplify your answer)?

(b) (5 points) How many permutations of $\mathbb{S}$ are there?

(c) (5 points) 2 letters were randomly chosen from $\mathbb{S}$. Determine the probability that there is a vowel among them.

(a) We have 2 types of 2-permutations. When two letters are the same, we have 3 2-permutations. When two letters are different, we have $6 \cdot 5 = 30$ permutations. Therefore, we have $30 + 3 = 33$.

(b) There are $P(2,2,2,1,1,1) = \frac{\mathbf{9!}}{\mathbf{(2!)^3}}$ permutations.

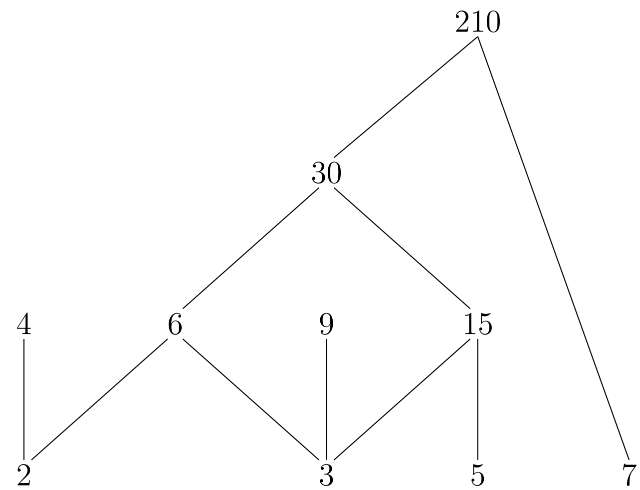
(c) Denote by E our event. $\overline{E}$ is the event of obtaining two consonants. So

$$P(\overline{E}) = \frac{\binom{6}{2}}{\binom{9}{2}} \qquad \text{and} \qquad P(E) = 1 - \frac{\binom{6}{2}}{\binom{9}{2}} = \frac{7}{12}.$$

First Name: ________________ Last Name: ________________

2. **On this page, only the answer will be graded. MARK YOUR ANSWER CLEARLY. But you do not need to justify your work.**

Consider the following partial order on the set $X = \{2, 3, 4, 5, 6, 7, 9, 15, 30, 210\}$: $x \leq_R y$ if and only if y is divisible by x (i.e. x is a divisor of y). Hasse diagram is drawn.



(a) (4 points) How many minimal elements are there?

4

(b) (4 points) Find the largest chain.

For example, 2, 6, 30, 210

(c) (4 points) Find the largest antichain.

For example, 4, 6, 9, 15, 7

(d) (4 points) Find the smallest chain partition.

For example
2, 4
3, 6, 30, 210
9
5,15
7

(e) (4 points) Find the smallest antichain partition.

For example
2, 3, 5, 7
4, 6, 9, 15
30
210

The Dilworth theorem states that if we found in (c) and (d) examples of the same length then this is really maximal antichain and smallest chain partition.

Dual to the Dilworth theorem states that if we found in (b) and (e) examples of the same length then this is really maximal chain and smallest antichain partition.

3. (a) (10 points) Construct permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ whose inversion sequence is 3, 5, 1, 2, 1, 2, 0, 0.

   7, 3, 5, 1, 4, 8, 2, 6

   (b) (5 points) Construct the inversion sequence of the permutation 7, 3, 5, 1, 4, 8, 2, 6.

   3, 5, 1, 2, 1, 2, 0, 0

4. (10 points) A chess player plans to train for his next match by playing at least one game each day for 90 days, but at most five games over any period of three consecutive days. Show that there must be some period of consecutive days during which he plays 29 games.

Let c_i be the number of games completed by the end of the ith day of training. Since at least one game is played each day, and at most five games are played in any three consecutive days, we have that $0 < c_1 < c_2 < \ldots < c_{90} \leq 150$. Now consider the increasing sequences of positive integers $c_1 < c_2 < \ldots < c_{90}$ and $c_1 + 29 < c_2 + 29 < \ldots < c_{90} + 29$. Between them they contain 180 numbers, all in the range from 1 to 179. Hence, by the pigeonhole principle, there are two of them that are equal. Since the numbers in each of the two sequences are all different, this pair must consist of one number from the first sequence and one number from the second sequence.

That is, for some integers r, s in the range from 1 to 90, we have $c_r = c_s + 29$. It follows that $c_r - c_s = 29$. Consequently that $s < r$, as $c_s < c_r$. Hence in the period of consecutive days from day $s + 1$ to day r the chess player completes exactly 29 games.

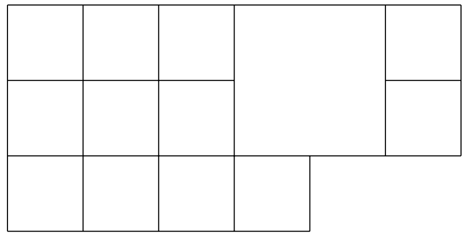
5. (10 points) Give an example of a noncyclic Gray code of order 3.

   000 − 001 − 011 − 010 − 110 − 100 − 101 − 111

6. (10 points) List all 3-subsets of $\{1, 2, 3, 4, 5, 6\}$ in the lexicographic order.

   123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456

7. (10 points) How many walks are there from the lower left corner to the upper right corner taking upwards and rightwards steps only? (note the excluded portions):



If the grid were complete, there would be $\binom{9}{3}$ routes from the lower left corner $(0,0)$ to the upper right $(6,3)$. We must remove those which pass through the excluded points $(4,2)$ or $(5,0)$ (we don't need to exclude $(6,0)$ since any path through $(6,0)$ would have to go through $(5,0)$ and thus already be excluded.

Exactly $\binom{6}{2} \cdot \binom{3}{1}$ paths pass through $(4,2)$. Likewise, exactly $\binom{5}{0} \cdot \binom{4}{1}$ paths go through $(5,0)$. No paths go through both points, so no overlaping, and we get there result

$$\binom{9}{3} - \binom{6}{2}\binom{3}{1} - \binom{5}{0}\binom{4}{1} = 35.$$

Another Solution. We have 2 types of paths.

Type 1: $(0,0) \to (3,3) \to (6,3)$. We have exactly $\binom{6}{3}$ such paths.

Type 2: $(0,0) \to (4,1) \to (5,1) \to (6,3)$. We have exactly $\binom{5}{1} \cdot \binom{3}{1}$ such paths.

$$\binom{6}{3} + \binom{5}{1}\binom{3}{1} = 35.$$

8. (10 points) Determine the number of 9-combinations of the multiset

$$\{1 \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}.$$

For a 9-combination in question, either it contains a or it does not. The number of 9-combination containing a is equal to the numbers of 8-combinations of $\{\infty \cdot b, \infty \cdot c, \infty \cdot d\}$, which comes to $\binom{8+3-1}{3-1} = \binom{10}{2}$. The number of 9-combinations that do not contain a is equal to the number of 9-combinations of $\{\infty \cdot b, \infty \cdot c, \infty \cdot d\}$, which comes to $\binom{9+3-1}{3-1} = \binom{11}{2}$. The answer is

$$\binom{10}{2} + \binom{11}{2} = 45 + 55 = 100.$$