Department of Mathematics, University of Wisconsin-Madison

Math 475 — Final Exam — Solutions — Fall 2023

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INSTRUCTIONS:

Time: **120 minutes**

Please write your name on every page.

Read the problems carefully and budget your time wisely.

No calculators or other electronic devices, please. **Turn off your phone.**

Please present your solutions in a clear manner. Justify your steps.
A numerical answer without explanation cannot get credit.
Cross out the writing that you do not wish to be graded on.

You must use correct notation to receive full credit.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	8	12	10	8	10	10	12	10	10	10	100

1. (8 points) In how many ways can 12 indistinguishable apples and 1 orange be distributed among 3 volleyball players in such a way that each player gets at least one fruit?

Solution: We need to hand out the orange (3 ways), give one apple to two other volleyball player, and then distribute remaining 10 apples to 3 players (balls and borders, $\binom{10+2}{2}$ ways). So answer is $3 \cdot \binom{12}{2} = 198$.

2. Let

$$S = \text{XXXYYYZZZZ}.$$

(There are three X's, three Y 's, and four Z's).

(a) (4 points) How many permutations of the letters of S are there?

(b) (4 points) How many permutations are there, if no two X's can be consecutive?

(c) (4 points) How many permutations are there, if all three of the Y's have to be consecutive?

Solution:

(a) This is given by the multinomial cofficient

$$\binom{10}{3,3,4} = 4200.$$

(b) Permute the Y's and Z's $\binom{7}{3}$ ways, and then insert the X's into the 8 possible spaces including the ends $\binom{8}{3}$ ways. So in total there are

$$\binom{7}{3} \cdot \binom{8}{3} = 1960.$$

(b) Other Solution. Let's consider $S'\{x, xx, 3 \cdot y, 4 \cdot y\}$. All permutations of S' are bad permutations for us. But permutations including xxx were counted twice. So

$$\binom{10}{3,3,4} - \binom{9}{1,1,3,4} + \binom{8}{1,3,4} = 1960.$$

(c) Glue all the Y 's together and permute X, X, X, YYY, Z, Z, Z, Z:

$$\binom{8}{3,1,4} = 280.$$

3. (10 points) Determine the number of permutations of $\{1,2,3,4,5,6,7,8,9\}$ in which no even numbers are in their natural positions.

Solution: Let the set S consist of the permutations of $\{1,2,3,4,5,6,7,8,9\}$. For $i \in \{2,4,6,8\}$ let A_i denote the set of permutations in S for which i is in its natural position. We seek $|\overline{A}_2 \cap \overline{A}_4 \cap \overline{A}_6 \cap \overline{A}_8|$. We have

set	size
S	$9!$
A_i	$8!$
$A_i \cap A_j$	$7!$
$A_i \cap A_j \cap A_k$	$6!$
$A_i \cap A_j \cap A_k \cap A_\ell$	$5!$

By inclusion/exclusion

$$|\overline{A}_2 \cap \overline{A}_4 \cap \overline{A}_6 \cap \overline{A}_8| = 9! - 4 \times 8! + 6 \times 7! - 4 \times 6! + 5! = 229080.$$

Another Solution:

$$D_9 + \binom{5}{1} D_8 + \binom{5}{2} D_7 + \binom{5}{3} D_6 + \binom{5}{4} D_5 + \binom{5}{5} D_4$$

4. (8 points) Write the generating function for the number of ways to make n dollars if you can only use 1 dollar, 5 dollar, 20 dollar, and 50 dollar bills.

Solution: This problems asked as about number of solutions of the equation

$$a + 5b + 20c + 50d = n$$

so

$$g(x) = \frac{1}{(1-x)(1-x^5)(1-x^{20})(1-x^{50})}.$$

5. (10 points) Solve the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2} + 9$$

for $n \geq 2$ with initial values $a_0 = 0$, $a_1 = 1$.

Solution:

Characteristic equation: $x^2 - 8x + 16 = 0 \Leftrightarrow (x-4)^2 = 0$. Plus root 1 from unhomogeneous part.

General solution: $a_4 = b \cdot 4^n + c \cdot n\dot{4}^n + d$.

Initial conditions:

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 = 8 + 9 = 17 \end{cases} \Leftrightarrow \begin{cases} b + d = 0 \\ 4b + 4c + d = 1 \\ 16b + 32c + d = 17 \end{cases} \Leftrightarrow \begin{cases} b = -1 \\ c = 1 \\ d = 1. \end{cases}$$

As a result

$$a_n = -4^n + n4^n + 1.$$

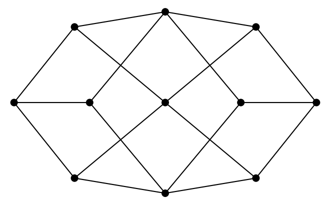
6. (10 points) How many non-isomorphic graphs on 12 vertices with degree sequence (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)? Describe all of them.

Solution: Such a graph must be collection of disjoint cycles. Knowing how many cycles and of each length is enough to tell the isomorphism type. This is the same as the number of partitions of 12 into parts of size at least 3. We have

$$\begin{aligned} 12 &= 12 \\ &= 9 + 3 \\ &= 8 + 4 \\ &= 7 + 5 \\ &= 6 + 6 \\ &= 6 + 3 + 3 \\ &= 5 + 4 + 3 \\ &= 4 + 4 + 4 \\ &= 3 + 3 + 3 + 3. \end{aligned}$$

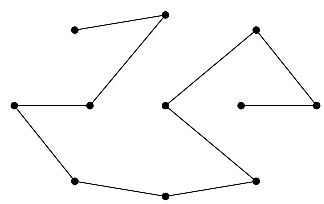
So there are 9 isomorphism types.

7. Let G be the graph below.
   (a) (6 points) Does G has the Hamiltonian path?
   (b) (6 points) Does G has the Hamiltonian cycle?

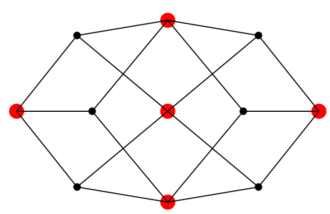


Solution:

(a) Yes. For example:

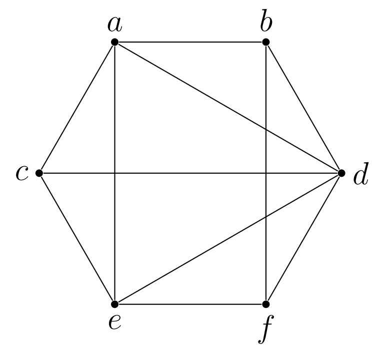


(b) Graph G is bipartite:

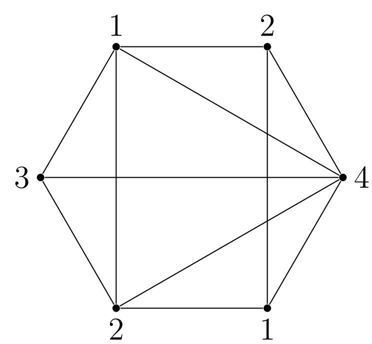


with sizes of parts 5 and 6. Then F does not have a Hamiltonian cycle.

8. (10 points) Find the chromatic number of the graph G below and proper coloring of $\chi(G)$ colors.



Solution: Consider induced subgraph $G|_{\{a,c,d,e\}}$. It is isomorphic to K_4, so $\chi(G) \geq 4$. On the other side there is 4-coloring (based on previous idea)



9. (10 points) Let G denote the group of symmetries for the regular 6-gon. Find the cycle index P_G.

Solution: The Group G is the dihedral group D_6 with 12 elements.The elements and monomials are

f	I	ρ	ρ^2	ρ^3	ρ^4	ρ^5	τ	$\rho \circ \tau$	$\rho^2\tau$	$\rho^3\tau$	$\rho^4\tau$	$\rho^5\tau$
$mon(f)$	z_1^6	z_6	z_3^2	z_2^3	z_3^2	z_6	z_2^3	$z_1^2z_2^2$	z_2^3	$z_1^2z_2^2$	z_2^3	$z_1^2z_2^2$

Therefore

$$P_G(z_1, \ldots z_6) = \frac{z_1^6 + 3z_1^2z_2^2 + 4z_2^3 + 2z_3^2 + 2z_6}{12}$$

10. (10 points) Find the number of necklaces that have 19 beads with colors red, white, and blue.

Solution: Label the bead locations clockwise $1, 2, \ldots, 19$. Let G denote the corresponding symmetry group. Then G is the dihedral group D_{19} with 38 elements. We have $G = \{\rho^i\}_{i=0}^{18} \cup \{\rho^i \circ \tau\}_{i=0}^{18}$ where ρ is the clockwise rotation through an angle of $360/n$ degrees, and τ is the reflection about the line of symmetry passing through bead 1. The cycle index of G is

$$P_G(z_1, z_2, \ldots z_n) = \frac{\sum_{f \in G} mon(f)}{|G|}.$$

For the identity $I \in G$ the corresponding monomial is z_1^{19}. For $1 \leq i \leq 18$ the monomial for ρ^i is z_{19}. For $0 \leq i \leq 18$ the monomial for $\rho^i \circ \tau$ is $z_1 z_2^9$. Therefore

$$P_G(z_1, z_2, \ldots z_n) = \frac{z_1^{19} + 19 z_1 z_2^9 + 18 z_{19}}{38}.$$

The number of necklaces is

$$P_G(3, 3, \ldots, 3) = \frac{3^{19} + 19 \cdot 3^{10} + 18 \cdot 3}{38}.$$