Department of Mathematics, University of Wisconsin-Madison

Math 467 — Exam 4 — Solutions — Fall 2023

NAME : (as it appears on Canvas)

EMAIL: @wisc.edu

PROFESSOR: MIKHAIL IVANOV

INSTRUCTIONS:

Time: **50 minutes**

Please write your name on every page.

No Calculators, No Notecards, No Notes

With the exception of the True/False question, and Short Answer question, you must justify your claims and use complete sentences in proofs.

You must use correct notation to receive full credit.

You can safely assume that all unknown quantitites in this exam, e.g. a, b, c, n, x, y, are always the integers.

Question:	1	2	3	4	5	6	7	Total
Points:	8	8	10	6	6	6	6	50

1. (8 points) For each statement below,choose true or false. You do not need to show your work. Please **fill in** the circle that corresponds to the correct answer.

   (a) $\sum_{d|n} \tau(d) = \sigma(n)$.

   ◯ TRUE

   ◯ FALSE

   Solution: False. $\sum_{d|n} d = \sigma(n)$

   (b) $\tau(n) = 5$ if and only if $n = p^4$.

   ◯ TRUE

   ◯ FALSE

   Solution: True. $(a_1 + 1) \dots (a_k + 1) = 5 \Leftrightarrow k = 1$ and $a_1 = 4$.

   (c) For all positive integer n holds $2^{\varphi(n)} \equiv 1 \pmod{n}$.

   ◯ TRUE

   ◯ FALSE

   Solution: False. It is true only for odd n.

   (d) Equation $\mu(n) = \mu(n + 1) = 1$ has solutions in positive integers.

   ◯ TRUE

   ◯ FALSE

   Solution: True. For example, $n = 14$, $n = 15$.

2. (8 points) **On this page, only the answer will be graded. MARK YOUR ANSWER CLEARLY. But you do not need to justify your work.**

   (a) Compute $\tau(120)$.

   > **Solution:** $\tau(120) = \tau(2^3)\tau(3)\tau(5) = 4 \cdot 2 \cdot 2 = 16.$

   (b) Compute $\sigma(250)$.

   > **Solution:** $\sigma(250) = \sigma(2)\sigma(5^3) = (2+1)(125+25+5+1) = 468.$

   (c) Compute $\mu(33)$.

   > **Solution:** $\mu(33) = \mu(3)\mu(11) = (-1)(-1) = 1.$

   (d) Compute $\varphi(200)$.

   > **Solution:** $\varphi(200)$=$\varphi(2^3)\varphi(5^2) = 2^2(2-1)5(5-1) = 80.$

3. (a) (5 points) Find the last two digits of 17^{882}.

Solution: Euler's Theorem gives us

$$17^{\varphi(100)} = 17^{40} \equiv 1 \pmod{100}.$$

So

$$17^{882} \equiv 17^2 = 289 \equiv 89 \pmod{100}.$$

(b) (5 points) Find the last two digits of 8^{222}.

Solution: We will find 8^{222} separately modulo 25 and modulo 4. Euler's Theorem gives us

$$8^{\varphi(25)} = 8^{20} \equiv 1 \pmod{25}.$$

So

$$8^{222} \equiv 8^2 = 64 \equiv 14 \pmod{25}.$$

Additionally we have $8^{222} \equiv 0 \pmod{4}$. Now, by CRT

$$\begin{cases} 8^{222} \equiv 14 \pmod{25} \\ 8^{222} \equiv 0 \pmod{4} \end{cases} \Leftrightarrow 8^{222} \equiv 64 \pmod{100}$$

4. (6 points) Compute the largest power of 48 that divides 50!

Solution: The key theorem is that the largest power of a prime p that divides an integer n is

$$\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$$

where $\left[n/p^k\right]$ is the greatest integer function. Since $48 = 2^4 \cdot 3$ we must find the largest power of 2 and the largest power of 3 that divides 50!. For 2, the computation is

$$[50/2] + [50/4] + [50/8] + [50/16] + [50/32] = 25 + 12 + 6 + 3 + 1 = 47.$$

Since $47 = 11 \cdot 4 + 3$, the largest power of 2^4 that divides 50! is thus $\left(2^4\right)^{11}$.

For 3 the computation is

$$[50/3] + [50/9] + [50/27] = 16 + 5 + 1 = 22.$$

Thus the largest power of 3 that divides 50! is 3^{22}.

It follows that largest power of 48 that divides 50! is 48^{11}.

5. (6 points) Derive the following expression for the alternating sum of the first $n \geq 2$ Fibonacci numbers:

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n+1} u_n = 1 + (-1)^{n+1} u_{n-1}.$$

Solution: First we establish the base case of $n = 2$. We have $0 = u_1 - u_2 = 1 + (-1)^{2+1} u_1 = 1 - 1$.

Next we turn to the induction step. We will assume that the statement holds for the n case:

Induction hypothesis is the n case:

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n+1} u_n = 1 + (-1)^{n+1} u_{n-1}.$$

We need to show the $n + 1$ case:

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n+1} u_n + (-1)^{n+2} u_{n+1} = 1 + (-1)^{n+2} u_n.$$

So we start with the lefthand side of the $n + 1$ case, which is

$$u_1 - u_2 + \ldots + (-1)^{n+1} u_n + (-1)^{n+2} u_{n+1} = \left[u_1 - u_2 + \ldots + (-1)^{n+1} u_n\right] + (-1)^{n+2} u_{n+1}$$

By the induction hypothesis, the sum in the brackets is known so we substitute:

$$= 1 + (-1)^{n+1} u_{n-1} + (-1)^{n+2} u_{n+1}$$

Now we do some algebra to rewrite this as

$$= 1 + (-1)^{n+2} (u_{n+1} - u_{n-1}) = 1 + (-1)^{n+2} u_n.$$

So we have shown that

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n+1} u_n + (-1)^{n+2} u_{n+1} = 1 + (-1)^{n+2} u_n,$$

which is what we needed to show.

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6. (6 points) For the Fibonacci sequence, establish the following:

$$u_{n+5} \equiv 3u_n \pmod{5}.$$

Solution:

Solution 1. $u_{n+5} = u_{n+4} + u_{n+3} = 2u_{n+3} + u_{n+2} = 3u_{n+2} + 2u_{n+1} = 5u_{n+1} + 3u_n \equiv 0 \pmod{5}$. (or you can use $u_{n+5} = u_n u_4 + u_{n+1} u_5$.)

Solution 2. Induction. Base $n = 1$ and $n = 2$ (double base)! Induction step: if $u_{n+5} \equiv 3u_n \pmod{5}$ and $u_{n+6} \equiv 3u_{n+1} \pmod{5}$ then $u_{n+7} = u_{n+6} + u_{n+5} \equiv 3u_{n+1} + 3u_n = 3u_{n+2}$.

Solution 3. Let's write sown u_n mod 5.

$$1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1$$

we see periodicity. Now consider $3u_n$ mod 5:

$$3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3, 0, 3, 3$$

We see that second sequence is first one shifted by 5.

7. (6 points) Let $t(n) = \tau(n) \cdot \varphi(n)$ and let $T(n) = \sum_{d|n} t(d)$. Prove that $T(n)$ is a multiplicative function.

Solution: We first observe that t is multiplicative, since if $\gcd(m, n) = 1$ then

$$t(mn) = \tau(mn)\varphi(mn) = \tau(m)\varphi(m)\tau(n)\varphi(n) = t(m)t(n)$$

where the middle equality is because τ and φ are multiplicative. Summing a multiplicative function over $\sum_{d|n}$ yields a new multiplicative function, and hence T is multiplicative as well.