Department of Mathematics, University of Wisconsin-Madison

Math 467 — Exam 2 — Fall 2023

NAME : (as it appears on Canvas)

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INSTRUCTIONS:

Time: **50 minutes**

Please write your name on every page.

No Calculators, No Notecards, No Notes

With the exception of the True/False question, you must justify your claims and use complete sentences in proofs.

You must use correct notation to receive full credit.

You can safely assume that all unknown quantitites in this exam, e.g. a, b, c, n, x, y, are always assumed to be the integers.

Question:	1	2	3	4	5	6	7	Total
Points:	8	6	6	6	6	6	12	50

1. (8 points) For each statement below, CIRCLE true or false. You do not need to show your work.

   (a) The product of any two numbers of the form $5n + 3$ is also of the form $5n + 3$.

   TRUE FALSE

   (b) Let $a, b \in \mathbb{Z}$. If $5a \equiv 5b \pmod{11}$ then $a \equiv b \pmod{11}$.

   TRUE FALSE

   (c) $1234 \equiv 1 - 2 + 3 - 4 \pmod{11}$.

   TRUE FALSE

   (d) There are exactly 5 primes between 30 and 50.

   TRUE FALSE

Solution.

(a) FALSE, it would have form $5n + 4$,

(b) TRUE, $\gcd(5, 11) = 1$,

(c) FALSE, it would be $4 - 3 + 2 - 1$,

(d) TRUE, 31, 37, 41, 43, 47.

2. (6 points) Compute the last digit of $(1!)^2 + (3!)^2 + (5!)^2 + (7!)^2 + \dots$.

   If $n \geq 5$ then $n! \equiv 0 \pmod{10}$, so

   $$(1!)^2 + (3!)^2 + (5!)^2 + (7!)^2 + \dots \equiv (1!)^2 + (3!)^2 \equiv 1 + 6^2 \equiv 7 \pmod{10}.$$

   Last digit is 7.

3. (6 points) Find some a, b, c such that $ac \equiv bc \pmod 8$ but $a \not\equiv b \pmod 8$.

   For example $a = 1$, $b = 2$, $c = 0$. Or $a = 0$, $b = 4$, $c = 2$.

4. (6 points) Find all solutions to the linear equation $5x + 7y = 11$. Express your answer in the form

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases} \qquad t \in \mathbb{Z}$$

for specific integers x_0, y_0, a, b.

$\gcd(5, 11) \mid 11$, so there are infinite amount of solutions. We should find one solution. Just by guess $(5, -2)$, or $(-2, 3)$, or from Euclidean algorithm: $1 = 5\dot{3} - 7 \cdot 2$, so $(33, -22)$. Now we can write formulas for the general solution:

$$\begin{cases} x = 33 - 7t \\ y = -22 + 5t \end{cases} \qquad t \in \mathbb{Z}$$

5. (6 points) Prove that $\sqrt{15}$ is not a rational number.

   Suppose, to the contrary, that $\sqrt{15}$ is a rational number, say, $\sqrt{15} = a/b$, where a and b are both integers with $\gcd(a, b) = 1$. Squaring, we get $a^2 = 15b^2$, so that $b \mid a^2$. If $b > 1$, then the Fundamental Theorem of Arithmetic guarantees the existence of a prime p such that $p \mid b$. It follows that $p \mid a^2$ and, so, $p \mid a$; hence, $\gcd(a, b) \geq p$. We therefore arrive at a contradiction, unless $b = 1$. But if this happens, then $a^2 = 15$, which is impossible $(3 < a < 4)$. Our supposition that $\sqrt{15}$ is a rational number is untenable, and so $\sqrt{15}$ must be irrational.

6. (6 points) Formulate and prove divisibility by 5 rule, i.e. how to determine which numbers are divisible by 5 using their decimal representation.

   Number is divisible by 5 if and only if it's last digit is 0 and 5.

   Proof. $\overline{a_n a_{n-1} \ldots a a_2 a_1 a_0} \equiv \overline{a_n a_{n-1} \ldots a a_2 a_1 0} + a_0 \equiv \overline{a_n a_{n-1} \ldots a a_2 a_1} \cdot 10 + a_0 \equiv a_0 \pmod{5}$.

7. (a) (6 points) Find any $k \geq 1$ such that $3^k \equiv 1 \pmod 7$.

$3^1 \equiv 3 \pmod 7$,

$3^2 \equiv 9 \equiv 2 \pmod 7$,

$3^3 \equiv 3 \cdot 2 \equiv 6 \equiv -1 \pmod 7$,

$3^6 \equiv (-1)^2 \equiv 1 \pmod 7$.

(b) (6 points) Find $0 \leq n \leq 6$ such that $3^{555} \equiv n \pmod 7$.

$3^6 \equiv 1 \pmod 7$, so $3^{6k} \equiv 1 \pmod 7$.

$555 = 6 \cdot 92 + 3$, so $3^{555} \equiv (3^6)^9 2 \cdot 3^3 \equiv -1 \equiv 6 \pmod 7$.