Department of Mathematics, University of Wisconsin-Madison

Math 467 — Exam 1 — Fall 2023

NAME : (as it appears on Canvas)

EMAIL: @wisc.edu

PROFESSOR:

INSTRUCTIONS:

Time: **50 minutes**

Please write your name on every page

No Calculators, No Notecards, No Notes
You must show your work and justify your answer,
with the exception of the True/False question to receive credit.
You must use correct notation to receive full credit.

Question:	1	2	3	4	5	Total
Points:	10	10	12	6	12	50

1. (10 points) For each statement below, CIRCLE true or false. You do not need to show your work.

   (a) For any integer $n \geq 1$, we have $\binom{2n+1}{2n} = 2n + 1$.

   TRUE FALSE

   (b) If we can write $2 = ar + bs$ with $r, s \in \mathbb{Z}$, then $2 = \gcd(a, b)$.

   TRUE FALSE

   (c) If $a \mid b$ and $b \mid c$ then $a \mid c$.

   TRUE FALSE

   (d) The greatest common divisor of a and b is the smallest positive integer that divides both a and b.

   TRUE FALSE

   (e) Every integer n can be written in the form $5k + 1$, $5k + 2$, $5k + 3$ or $5k + 4$ for some integer k.

   TRUE FALSE

2. (10 points) Prove that

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$

for all $n \geq 1$.

3.

(a) (6 points) Use the Euclidean Algorithm to compute $\gcd(81, 30)$. Show your work clearly!

(b) (6 points) Using your work in part (a), find integers s, t such that $\gcd(81, 30) = s \cdot 81 + t \cdot 30$. Show your work clearly!

4. (6 points) Use the binomial theorem to show that

$$\binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + \ldots + 3^n\binom{n}{n} = 4^n.$$

5. Prove that

(a) (6 points) For any integer a, the product $a(a-7)(a+4)$ is divisible by 3.

(b) (6 points) If $a \mid b$ and $a \mid c$ then $a^2 \mid bc$.