Department of Mathematics, University of Wisconsin-Madison

Math 467 — Exam 1 — Fall 2023

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PROFESSOR:

INSTRUCTIONS:

Time: **50 minutes**

Please write your name on every page

No Calculators, No Notecards, No Notes
You must show your work and justify your answer,
with the exception of the True/False question to receive credit.
You must use correct notation to receive full credit.

Question:	1	2	3	4	5	Total
Points:	10	10	12	6	12	50

1. (10 points) For each statement below, CIRCLE true or false. You do not need to show your work.

   (a) For any integer $n \geq 1$, we have $\binom{2n+1}{2n} = 2n + 1$.

   TRUE FALSE

   (b) If we can write $2 = ar + bs$ with $r, s \in \mathbb{Z}$, then $2 = \gcd(a, b)$.

   TRUE FALSE

   (c) If $a \mid b$ and $b \mid c$ then $a \mid c$.

   TRUE FALSE

   (d) The greatest common divisor of a and b is the smallest positive integer that divides both a and b.

   TRUE FALSE

   (e) Every integer n can be written in the form $5k + 1$, $5k + 2$, $5k + 3$ or $5k + 4$ for some integer k.

   TRUE FALSE

Solution: (a): True; (b): False (the gcd could be 1 or 2); (c): True. (d): False. It is the LARGEST positive integer that divides both a and b. (e): False. We are missing the case $n = 5k$.

2. (10 points) Prove that

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$

for all $n \geq 1$.

Solution: First we establish the base case of $n = 1$. We have $1^2 = \frac{1 \cdot 1 \cdot 3}{3}$.

Next we turn to the induction step. We will assume that the statement holds for the $n-1$ case:

Induction hypothesis is the n case:

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$

We need to show the $n+1$ case:

$$1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}.$$

So we start with the lefthand side of the $n+1$ case, which is

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 + (2n+1)^2 = \left[1^2 + 3^2 + 5 + \cdots + (2n-1)^2\right] + (2n+1)^2$$

By the induction hypothesis, the sum in the brackets is known so we substitute:

$$= \frac{n(2n-1)(2n+1)}{3} + (2n+1)^2$$

Now we do some algebra to rewrite this as

$$= \frac{2n+1}{3}(2n^2 - n + 3(2n+1)) = \frac{(n+1)(2n+1)(2n+3)}{3}.$$

So we have shown that

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3},$$

which is what we needed to show.

3.

(a) (6 points) Use the Euclidean Algorithm to compute $\gcd(81, 30)$. Show your work clearly!
Solution:

$$\begin{aligned} 81 &= 30 \cdot 2 + 21 \\ 30 &= 21 \cdot 1 + 9 \\ 21 &= 9 \cdot 2 + 3 \\ 9 &= 3 \cdot 3 + 0 \end{aligned}$$

$\boxed{\text{So } \gcd(81, 30) = 3.}$

(b) (6 points) Using your work in part (a), find integers s, t such that $\gcd(81, 30) = s \cdot 81 + t \cdot 30$. Show your work clearly!
Now we substitute:

$$\begin{aligned} 3 &= 21 - 2 \cdot 9 \\ &= 21 - 2 \cdot (30 - 21) \\ &= 3 \cdot 21 - 2 \cdot 30 \\ &= 3 \cdot (81 - 2 \cdot 30) - 2 \cdot 30 \\ &= 3 \cdot 81 - 8 \cdot 30 \end{aligned}$$

So $s = 3$ and $t = -8$ work and $\boxed{3 = 3 \cdot 81 - 8 \cdot 30.}$

4. (6 points) Use the binomial theorem to show that

$$\binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + \ldots + 3^n\binom{n}{n} = 4^n.$$

Solution: By the Binomial Theorem, we have

$$4^n = (3+1)^n = \sum_{k=0}^{n} \binom{n}{k} 3^k 1^{n-k}$$

Simplifying the righthand side, by omitting the coefficient of 1^{n-k} and rearranging the terms yields

$$\sum_{k=0}^{n} 3^n \binom{n}{k} = \binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + \ldots + 3^n\binom{n}{n},$$

which completes the proof.

5. Prove that

(a) (6 points) For any integer a, the product $a(a-7)(a+4)$ is divisible by 3.

Solution: We can write a in one of the forms $a = 3k$ or $a = 3k+1$ or $a = 3k+2$. We now prove the statement by cases.

Case 1: If $a = 3k$ then $3|a$ and thus $3|a(a-7)(a+4)$.

Case 2: If $a = 3k+1$ then $a-7 = 3k-6 = 3(k-2)$ is divisble by 3 and thus $3|a(a-7)(a+4)$.

Case 3: If $a = 3k+2$ then $a+4 = 3k+6 = 3(k+2)$ and is thus divisible by 3 and thus $3|a(a-7)(a+4)$.

(b) (6 points) If $a \mid b$ and $a \mid c$ then $a^2 \mid bc$.

Solution: Since $a|b$, we have $b = ma$ for some integer m. Since $a|c$ we have $c = na$ for some integer n. Now $bc = (ma)(na) = (mn)a^2$. Thus bc is a multilple of a^2 and so $a^2|bc$ as claimed.