Midterm Exam 2

1. Suppose that the circumcenter and incenter of $\triangle ABC$ are the same point. Prove that the triangle is equilateral.

   Join O to vertices A and B and note that $\triangle OAB$ is isosceles since we know that $OA = OB$. By the pons asinorum, we have $\angle OAB = \angle OBA$. Since we are assuming that O is also the incenter, we know that OA and OB bisect angles A and B, and we conclude that $\angle A = 2\angle OAB$ and $\angle B = 2\angle OBA$. It follows that $\angle A = \angle B$, and so by the converse of the pons asinorum, we see that $CA = CB$. Since we could have started with any two of the three vertices, it follows that all of the sides are equal, as required.

2. Prove that the Euler line of the medial triangle of $\triangle ABC$ is the same as the Euler line of $\triangle ABC$.

   One way to prove that 2 lines are coincide is to find 2 common points. Actually we can provide 3 (by any 2 is enough).

   The centroid of $\triangle ABC$ lie on the Euler line and coincide with the medial centroid (Exercise G.6a).

   The center of 9-point circle of $\triangle ABC$ lie on the Euler line and coincide with the medial circumcenter (End of section 2.4).

   The circumcenter of $\triangle ABC$ lie on the Euler line and coincide with the medial orthocenter (Exercise G.6b).

3. Show that for any triangle with inradius r, circumradius R and exradii r_a, r_b, r_c holds

   $$r_a + r_b + r_c = 4R + r.$$

   Let a, b, c, s, $\mathcal{A}$ be sides, semi-perimeter and area of out triangle. Using formulas for r, R, r_a, r_b, r_c:

   $$\frac{A}{s-a} + \frac{A}{s-b} + \frac{A}{s-c} = 4\frac{abc}{4\mathcal{A}} + \frac{A}{s}$$

   multiply by $\mathcal{A}$ and use Heron's formula:

   $$s(s-b)(s-c) + s(s-a)(s-c) + s(s-a)(s-b) = abc + (s-a)(s-b)(s-c)$$

   Now we can expand everything and check. To simplify computations, we will introduce new variable:

   $$x = s-a, \qquad y = s-b, \qquad z = s-c.$$

   In this new variables

   $$s = x+y+z, \qquad a = y+z, \qquad b = x+z, \qquad c = x+y,$$

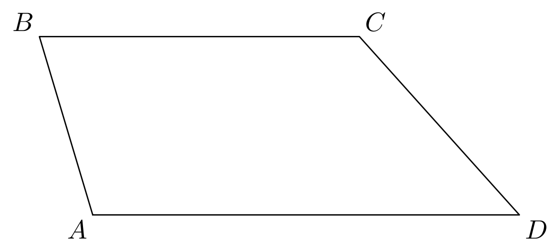
   and our identity

   $$(x+y+z)yz + (x+y+z)xz + (x+y+z)xy = (y+z)(x+z)(x+y) + xyz$$

   which is easy to check.

4. Given trapezoid with bases 12 and 16 and legs 7 and 9. Find angles of the trapezoid.

   Let $ABCD$ be the trapezoid with bases $AB = 7$, $BC = 12$, $CD = 9$, $DA = 16$. Let $\angle BAD = \alpha$ and $\angle CDA = \beta$, so $\angle ABC = 180° - \alpha$, $\angle BCD = 180° - \beta$.



   Solution 1. Draw the diagonal BD and apply The Law of Cosines for both triangles:

$$BD^2 = 7^2 + 16^2 - 2 \cdot 7 \cdot 16 \cdot \cos\alpha$$
$$BD^2 = 9^2 + 12^2 - 2 \cdot 9 \cdot 12 \cdot \cos(180° - \beta)$$

   So

$$7^2 + 16^2 - 2 \cdot 7 \cdot 16 \cdot \cos\alpha = 9^2 + 12^2 + 2 \cdot 9 \cdot 12 \cdot \cos\beta.$$

   Similarly we can draw the diagonal AC and obtain second equation for $\cos\alpha$ and $\cos\beta$:

$$7^2 + 12^2 + 2 \cdot 7 \cdot 12 \cdot \cos\alpha = 9^2 + 16^2 - 2 \cdot 9 \cdot 16 \cdot \cos\beta.$$

   We can solve linear system and find:

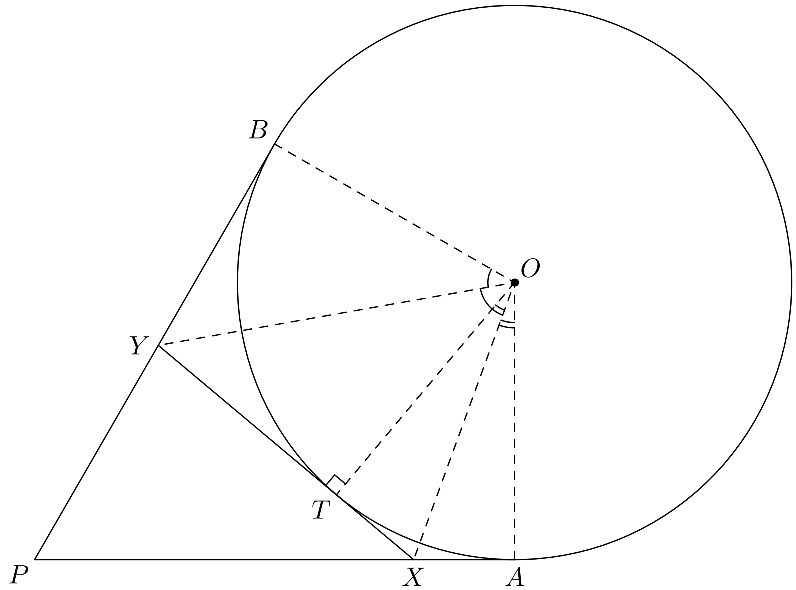
$$\begin{cases} \cos\alpha = -\dfrac{2}{7} \\ \cos\beta = \dfrac{2}{3}. \end{cases}$$

   Solution 2. Drop perpendicular BH from B to the line AD and let $AH = x$. We will think that x is positive if H is to the right of A and negative otherwise. Drop perpendicular CK from C to the line AD and let $DK = y$. We will think that y is positive if H is to the left of D and negative otherwise.

   For all cases $HK = 12$, $x + y$=4 ($AD = x + HK + y$) and $BH = CK$. So, apply Pythagorean Theorem for $\triangle ABH$ and $\triangle CDK$ we have:

$$\begin{cases} 49 - x^2 = 81 - y^2 \\ x + y = 4 \end{cases} \Leftrightarrow \begin{cases} x = -2 \\ y = 6 \end{cases} \Leftrightarrow \begin{cases} \cos\alpha = \dfrac{x}{7} = -\dfrac{2}{7} \\ \cos\beta = \dfrac{y}{9} = \dfrac{2}{3}. \end{cases}$$

5. Lines PA and PB are tangent to a circle centered at O; let A and B be the tangent points. A third tangent to the circle is drawn; it intersects with segments PA and PB at points X and Y, respectively. Prove that the value of angle XOY does not depend on the choice of the third tangent.

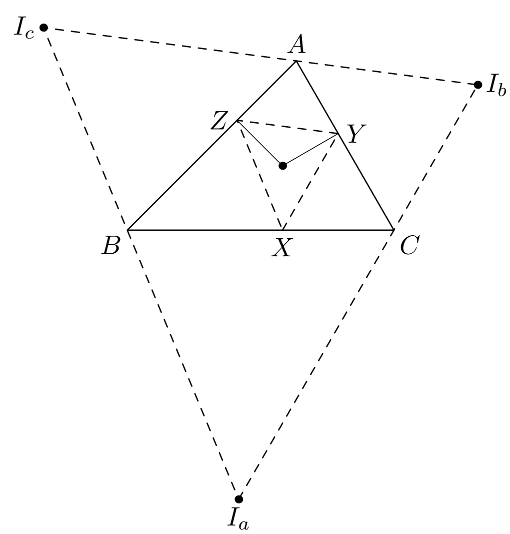


Look at the picture. Let T be the tangent point of XY. We have $\angle BOY = \angle YOT$ and $\angle TOX = \angle XOA$, so

$$\angle XOY = \frac{1}{2}\angle BOA,$$

which is independent from the position of the line XY.

6. Let I_a, I_b, I_c denote the centers of the exscribed circles correspond ing to vertices A, B and C respectively for $\triangle ABC$. Let X, Y, Z denote the points where the incircle of $\triangle ABC$ meets the three sides of $\triangle ABC$. Prove that $\triangle XYZ \sim \triangle I_aI_bI_c$.



Solution 1. We will show that corresponding sides are parallel. Let $\angle BAC = \alpha$. Triangle $\triangle AZY$ is isosceles, so $\angle AZY = \frac{1}{2}(180^\circ - \alpha)$. On the other side, AI_a is exterior angle bisector, so $\angle ZAI_c = \angle BAI_c = \frac{1}{2}(180^\circ - \alpha)$. So, $ZY \parallel I_cI_b$.

Solution 2. We will find corresponding angles of two triangles. Let $\angle BAC = \alpha$ and $\angle ABC = \beta$. $\triangle AZY$ is isosceles, so $\angle AZY = \frac{1}{2}(180^\circ - \alpha)$. Similarly $\angle BZX = \frac{1}{2}(180^\circ - \beta)$, so $\angle XZY = \frac{1}{2}(\alpha + \beta)$.

On the other side, AI_a is exterior angle bisector, so $\angle BAI_c = \frac{1}{2}(180^\circ - \alpha)$. Similarly $\angle ABI_c = \frac{1}{2}(180^\circ - \beta)$, so $\angle AI_cB = \angle I_bI_cI_a = \frac{1}{2}(\alpha + \beta)$.