Midterm Exam 1 — Solutions

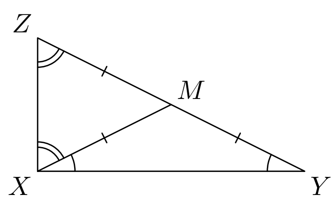
1. In quadrilateral $ABCD$, suppose that $\angle A = \angle C$ and $\angle B = \angle D$. (We are referring to the interior angles, of course.) Show that $ABCD$ is a parallelogram.

   We know that the sums of the interior angles of a quadrilateral is $360°$. Therefore, $\angle A+\angle B+\angle C+\angle D = 360°$. Since $\angle A = \angle C$ and $\angle B = \angle D$, it follows that $2(\angle A+\angle B) = 360°$. Therefore, $\angle A+\angle B = 180°$. In other words, $\angle A$ and $\angle B$ are supplementary. Extending DA past A to point X, we have $\angle XAB = \angle B$. We thus have congruent interior alternate angles on the transversal BX, so $AD \parallel BC$. Similarly, $\angle A + \angle D = 180°$, so $AB \parallel CD$ and $ABCD$ is a parallelogram.

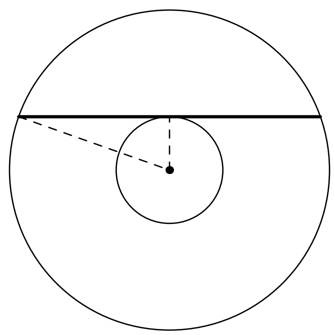
2. Let $\triangle XYZ$ be any triangle with median equal to the half of opposite side. Show that $\triangle XYZ$ is a right triangle.

   Let median XM be equal the half of YZ. We have $MX = MY = MZ$. So, triangles $\triangle MXY$ and $\triangle MXZ$ are isosceles and we have

   $$180° = \angle X + \angle Y + \angle Z = \angle MXY + \angle MXZ + \angle Y + \angle Z \Leftrightarrow \angle X = \angle MXY + \angle MXZ = 90°.$$



3. A chord AB of a circle with center O is tangent to a smaller circle with center O. Assuming that $AB = 12$, determine the area of the annular region between two circles.

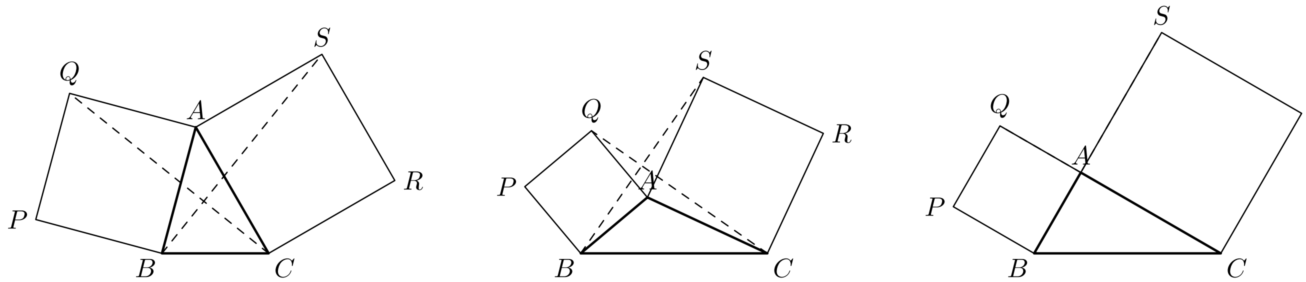


   Let r and R be radii of small and large circles. We have only one given measurement (chord), so we need to find algebraic connection between r, R and 12. Radius to the tangetn point is perpendicular to the tangent, so we have right angled triangle and $R^2 - r^2 = 6^2$.

   Now consider area between two circles. Obviously it is the differences between πR^2 and πr^2, so

   $$\mathcal{A} = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = 36\pi.$$

4. Given $\triangle ABC$, we construct squares $ABPQ$ and $ACRS$ outward from $\triangle ABC$ as shown. Prove that $CQ = BS$. [**Hint:** There are different cases based on the measure of $\angle A$.]



If we mark equal seqments, we will see that our two segments are side of $\triangle QAC$ and $\triangle BAS$. We have $QA = BA$ and $AC = AS$. Angles depends on configuration.

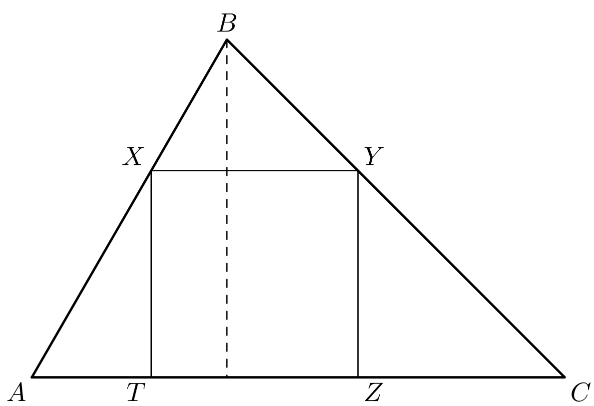
Case 1: $\angle QAC = 90° + \angle BAC = \angle BAC + 90° = \angle BAS$.

Case 2: $\angle QAC = 360° - 90° - \angle BAC = \angle BAS$.

In both cases $\triangle QAC \cong \triangle BAS$ and $QC = BS$ as corresponding elements.

In the special case 3 ($\angle BAC = 90$) we have just $QC = QA + AC = BA + AS = BS$.

5. A square is inscribed in the triangle $\triangle ABC$ such that two vertices lie on side AC and 2 others lie on AB and BC. Find side of the square, if $AC = a$ and the altitude from B is h.



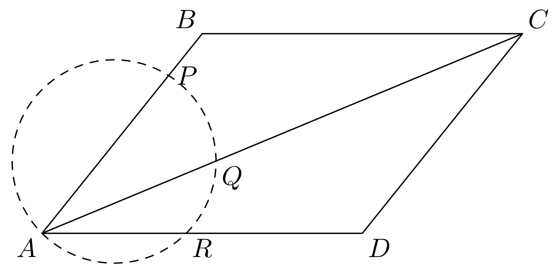
Solution 1: Let x be the side of the square. Consider $\triangle BXY$. The altitude from B is equal to $h - x$, base is equal x and $\triangle BXY \sim \triangle ABC$. So

$$\frac{h}{a} = \frac{h-x}{x} \Leftrightarrow x = \frac{ah}{a+h}.$$

Solution 2: Let x be the side of the square and $AT = y$.

$$\frac{ah}{2} = \mathcal{A}_{ABC} = \mathcal{A}_{ATX} + \mathcal{A}_{BXY} + \mathcal{A}_{YZX} + \mathcal{A}_{XYZT} = \frac{yx}{2} + \frac{(h-x)x}{2} + \frac{(a-x-y)x}{2} + x^2 = \frac{hx+ax}{2} \Leftrightarrow x = \frac{ah}{a+h}.$$

6. Given parallelogram $ABCD$, suppose a circle through vertex A intersects side AB at P, diagonal AC at Q, and side AD at R as show in the figure below.



Show that $\triangle PQR \sim \triangle CBA$.

Just angle chasing using inscribed quadrilateral $APQR$.

$$\angle BAC = \angle PAQ = \angle PRQ$$

and

$$\angle BCA = \angle CAD = \angle QAR = \angle QPR.$$

So, $\triangle PQR \sim \triangle CBA$ by AA.