Final Exam — Solutions

1. Recall that the Gergonne point of a triangle is the point of intersection of the three segments joining the vertices of the triangle with points of tangency of the incircle with the opposite side. Let T be the Gergonne point of $\triangle ABC$. Show that if T coincides with the **incenter** of $\triangle ABC$, then the triangle must be equilateral.

   Let P be the point of tangency on the side BC. By definition of point T we know that $A - T - P$ is one line, so $A - I - P$ is one line, but $IP \perp BC$, so $AP \perp BC$, so angle bisector is coincide with altitude, so $AB = AC$. Similarly $AB = BC$, so $\triangle ABC$ is equilateral.

2. In quadrilateral $ABCD$, suppose that $AB \parallel CD$ and $\angle B = \angle D$. Show that $ABCD$ is a parallelogram.

   Extending DA past A to point X, we have $\angle D = \angle XAB$ as corresponding angles. We thus have congruent interior alternate angles on the transversal AB, so $AD \parallel BC$. And we know $AB \parallel CD$, so $ABCD$ is a parallelogram.

3. Find angle between vectors $\vec{a}$ and $\vec{b}$ if $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\left(\vec{a} - \vec{b}\right)^2 + \left(2\vec{a} - \vec{b}\right)^2 = 56$.

   $56 = \left(\vec{a} - \vec{b}\right)^2 + \left(2\vec{a} - \vec{b}\right)^2 = \vec{a}^2 - 2\vec{a}\,\vec{b} + \vec{b}^2 + 4\vec{a}^2 - 4\vec{a}\,\vec{b} + \vec{b}^2 = 4+9+16+9-6\vec{a}\,\vec{b} \Leftrightarrow \vec{a}\,\vec{b} = -3$
   Now we can find angle between vectors:

   $$\cos\alpha = \frac{\vec{a}\,\vec{b}}{|\vec{a}| \cdot |\vec{b}|} = -\frac{1}{2},$$

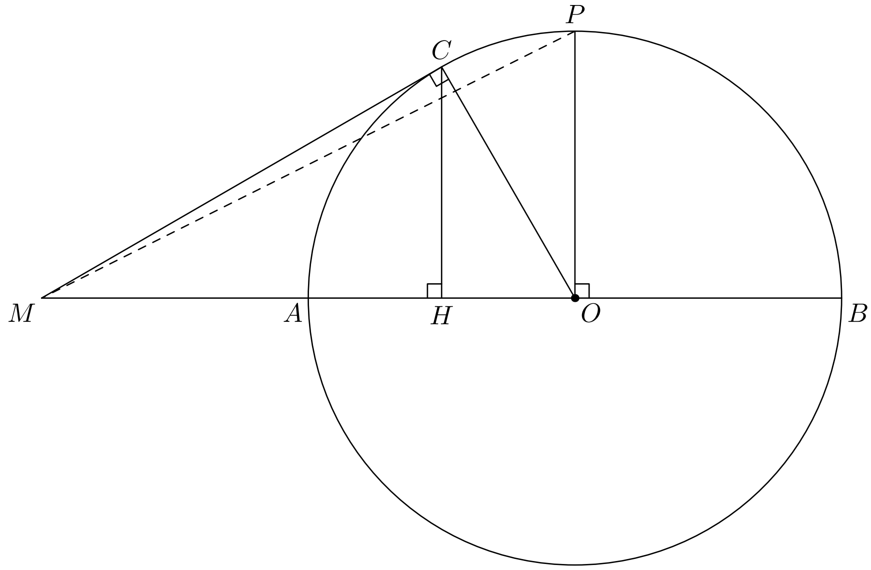
   so angle is $120°$.

4. Show that cevians bisecting perimeter of the triangle are concurrent.

   Let a, b, c be sides of the triangle. Let cevian to the side c divide it in parts x and y. We have $x + y = c$, $b + x = a + y = \frac{1}{2}(a + b + c)$, so $x = \frac{1}{2}(c + a - b)$, $y = \frac{1}{2}(c + b - a)$. Now we can compute such lengths for all cevians and check cevian s product to use Ceva's Theorem:

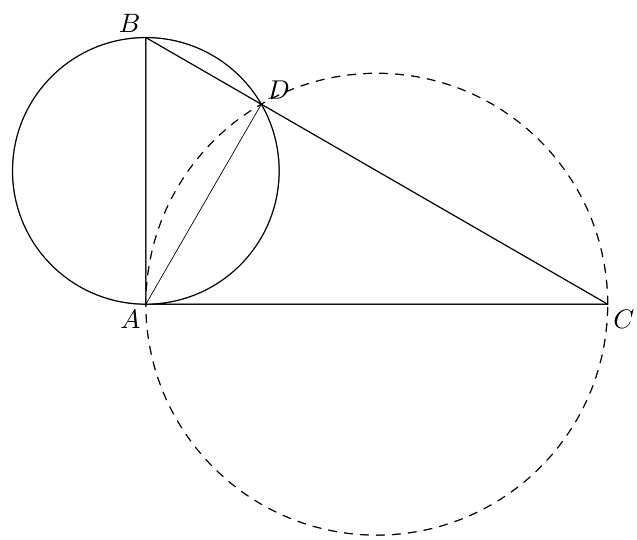
   $$\frac{\frac{1}{2}(c + a - b)}{\frac{1}{2}(c + b - a)}\,\frac{\frac{1}{2}(a + b - c)}{\frac{1}{2}(a + c - b)}\,\frac{\frac{1}{2}(b + c - a)}{\frac{1}{2}(b + a - c)} = 1.$$

5. Let M be a point outside of circle ω with center O. The line OM intersects circle ω at points A and B with $MA = a$, $MB = b$. Tangent line from M touch ω at point C. Point H is the projection of point C on AB. Perpendicular line from O to AB intersects ω at P. Find lengths MO, MP, MC and MH.



$AB = b-a$, so $r = OA = OP = \frac{1}{2}(b-a)$ and $MO = b+\frac{1}{2}(b-a) = \frac{1}{2}(a+b)$. $MP = \sqrt{OM^2+OP^2} = \sqrt{\frac{a^2+b^2}{2}}$. $MC = \sqrt{MO^2-CO^2} = \sqrt{ab}$. Last thing is CH. We can use similarity, or just area: $CH \cdot MO = 2\mathcal{A}_{MOC} = MC \cdot CO \Leftrightarrow CH = \frac{\sqrt{ab}\frac{1}{2}(b-a)}{\frac{1}{2}(a+b)}$, so $MH = \sqrt{MC^2-CH^2} = \sqrt{ab - \frac{ab(b-a)^2}{(a+b)^2}} == \frac{\sqrt{ab}\sqrt{(a+b)^2-(a-b)^2}}{a+b} = \frac{2ab}{a+b}$.

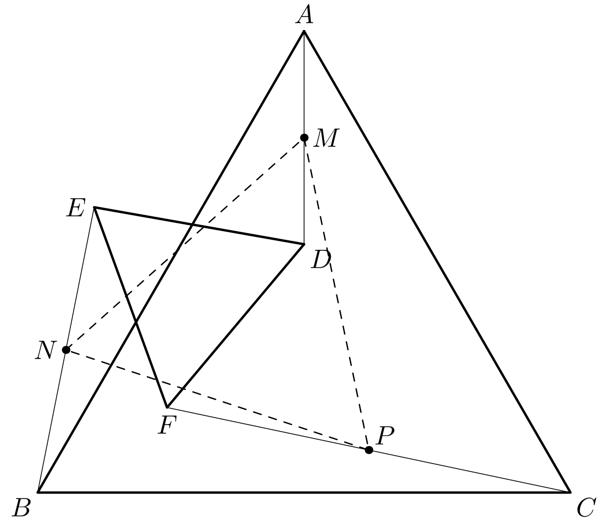
6. Let ω be a circle, having an arm of right angled triangle as a diameter. The circle cut the hypotenuse at ratio $1:3$. Find angles of the right angled triangle.



Let $\triangle ABC$ be our triangle, and D be the point of intersection of hypotenuse and the circle. segment AB is the diameter, so $\angle ADB = 90°$, so D is the feet of altitude. It explain, why order of ratio (1:3 or 3:1) is not important.

We know that the altitude divide the hypotenuse at segments x and $3x$. $AD = \sqrt{BD \cdot DC} = \sqrt{3}x$, and $AB = 2x$, $AC = 2\sqrt{3}x$, so angles are $30°$ and $60°$.

7. Given two equilateral triangles $\triangle ABC$ and $\triangle DEF$ and points M, N, P: $MA = MD$, $NB = NE$ and $PC = PF$. Prove that $\triangle MNP$ is an equilateral triangle.



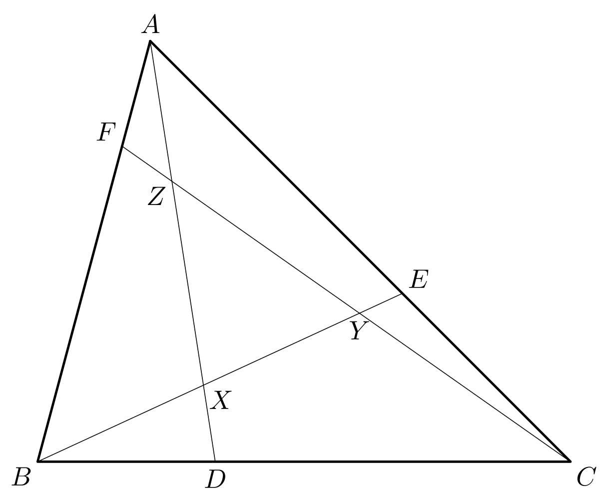
Let $\mathbb{T}$ be a $60°$ counterclockwise rotation. We have $\mathbb{T}(\overrightarrow{CA}) = \overrightarrow{CB}$, $\mathbb{T}(\overrightarrow{FD}) = \overrightarrow{FE}$. Now

$$\mathbb{T}(\overrightarrow{PM}) = \mathbb{T}\Big(\frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{FD}\Big) = \frac{1}{2}\mathbb{T}\Big(\overrightarrow{CA}\Big) + \frac{1}{2}\mathbb{T}\Big(\overrightarrow{FD}\Big) = \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{FE} = \overrightarrow{PN}.$$

(We used usual identity for 2 midpoints: $\overrightarrow{PM} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{FD}$ and $\overrightarrow{PN} = \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{FE}$). So, $\triangle PMN$ is equilateral too.

8. In the triangle ABC, let D, E and F are points on sides BC, AC, and AB, respectively. Let $AF/FB = r$, $BD/DC = s$, and $CE/EA = t$. Let X be the intersection of AD and BE, Y be the intersection of BE and CF, and Z be the intersection of CF and AD. Show, that

$$\frac{\mathcal{A}_{XYZ}}{\mathcal{A}_{ABC}} = \frac{(rst-1)^2}{(rs+r+1)(st+s+1)(tr+t+1)}.$$



Apply the Menelaus' Theorem for $\triangle ADB$ and points Z, C and F.

$$\frac{AZ}{ZD}\,\frac{DC}{CB}\,\frac{BF}{FA} = 1 \Leftrightarrow \frac{AZ}{ZD} = (s+1)r \Leftrightarrow \frac{AZ}{AD} = \frac{sr+r}{sr+r+1}.$$

Now

$$\frac{\mathcal{A}_{AZC}}{\mathcal{A}_{ABC}} = \frac{AZ}{AD}\,\frac{DC}{BC} = \frac{sr+r}{sr+r+1}\,\frac{1}{s+1} = \frac{r}{rs+r+1}.$$

And similarly other two:

$$\frac{\mathcal{A}_{CYB}}{\mathcal{A}_{ABC}} = \frac{s}{st+s+1}, \qquad \frac{\mathcal{A}_{CYB}}{\mathcal{A}_{ABC}} = \frac{t}{tr+t+1}.$$

Now

$$\frac{\mathcal{A}_{XYZ}}{\mathcal{A}_{ABC}} = 1 - \frac{r}{rs+r+1} - \frac{s}{st+s+1} - \frac{t}{tr+t+1}.$$

Do we want to simplify it? No. We want to say that denominator will be $(rs+r+1)(st+s+1)(tr+t+1)$, and numeratot will be some polynomial of degree 6 with leading term $r^2s^2t^2$. This polynomial are always positive and equal 0 if $rst = 1$ (Ceva's theorem)), so it is just $(rst-1)^2$.