1. Let $|\vec{a}| = 2$, $|\vec{b}| = 3$, $\angle(\vec{a}, \vec{b}) = 60^\circ$. Find $|2\vec{b} - \vec{a}|$.

**Solution.**

$$|2\vec{b} - \vec{a}| = \sqrt{(2\vec{b} - \vec{a})^2} = \sqrt{4(\vec{b})^2 - 4 \vec{b} \cdot \vec{a} + (\vec{a})^2} = \sqrt{4 \cdot 9 - 4 \cdot 3 \cdot 2 \cdot \cos(60^\circ) + 4} = \sqrt{36 - 12 + 4} = 2\sqrt{7}.$$

2. Recall that the Gergonne point of a triangle is the point of intersection of the three segments joining the vertices of the triangle with points of tangency of the incircle with the opposite side. Let $T$ be the Gergonne point of $\triangle ABC$. Show that if $T$ coincides with the orthocenter of $\triangle ABC$, then the triangle must be equilateral.

![Diagram of triangle ABC with points T and H, and line TP perpendicular to BC.]

**Solution.** Let $P$ be the point of tangency on the side $BC$. By definition of point $T$ we know that $A - T - P$ is one line, so $A - H - P$ is one line, but $AH \perp BC$, so $TP = HP \perp BC$, but $IP \perp BC$, so $I$ lie on the altitude. Now we have altitudes coincide with angle bisectors, so $\triangle ABC$ is equilateral.

3. In the following figure, $\triangle ABC$ is a triangle, $P$ and $Q$ are points on $BC$ and $CA$ respectively, and $R$ is the point of intersection of $AP$ and $BQ$. Given $AQ/AC = 1/2$, and $AR/AP = 3/8$, find $BP/BC$.

![Diagram of triangle ABC with points R, Q, and P.]

**Solution.** Apply Menelaus’ Theorem for the triangle $APC$ and points $R$, $B$ and $Q$. We have

$$\frac{AR}{RP} \cdot \frac{PB}{BC} \cdot \frac{CQ}{QA} = 1 \iff \frac{3}{5} \cdot \frac{PB}{BC} \cdot \frac{1}{1} = 1 \iff \frac{PB}{BC} = \frac{5}{3}.$$

4. Let $ABCDEF$ be the regular hexagon with $AB = 1$. Find $\overrightarrow{BA} \cdot \overrightarrow{CE}$.
Solution 1. \( \overrightarrow{BA} \cdot \overrightarrow{CE} = \overrightarrow{BA} \cdot \overrightarrow{BF} = \overrightarrow{BA} \cdot \overrightarrow{BA} + \overrightarrow{BA} \cdot \overrightarrow{AF} = 1^2 + 1 \cdot 1 \cdot \cos(60^\circ) = 1.5 \).

Solution 2. \( \overrightarrow{BA} \cdot \overrightarrow{CE} = (\overrightarrow{OA} - \overrightarrow{OB}) \cdot (\overrightarrow{OE} - \overrightarrow{OC}) = \overrightarrow{OA} \cdot \overrightarrow{OE} - \overrightarrow{OA} \cdot \overrightarrow{OC} - \overrightarrow{OB} \cdot \overrightarrow{OE} + \overrightarrow{OB} \cdot \overrightarrow{OC} = \cos 120^\circ - \cos 120^\circ - \cos 180^\circ + \cos 60^\circ = 1.5 \).

Solution 3. Cartesian coordinates.

5. Given any nonequilateral \( \triangle ABC \), identify whether each of the following points associated with \( \triangle ABC \) always lies on its Euler line, its nine-point circle, both, or neither.

1. The orthocenter \( H \). (The Euler line)
2. The Euler points \( X \), \( Y \), and \( Z \). (The 9-point circle)
3. The incenter \( I \). (neither)
4. The midpoint \( Q \) of side \( AC \). (The 9-point circle)
5. The nine-point center \( N \). (The Euler line)
6. The centroid \( G \). (The Euler line)
7. The foot \( D \) of the altitude from \( A \). (The 9-point circle)
8. The circumcenter \( O \). (The Euler line)
9. The vertex \( B \). (neither)
10. The Gergonne point. (neither)

6. What is the length of the bold segment \( x \)? (three arcs are semicircles of diameter 10, the segment is the tangent to the right circle)?

Solution. We have right-angled triangle \( DCF \), with hypotenuse \( DF = 25 \), and similar triangle \( EHF \) with ratio of similarity \( \frac{15}{25} \), so \( EH = \frac{3}{5} DC = 3 \). No we have isosceles triangle \( AEB \) with altitude equals 3 and sides equal 5. We use Pythagorean and have \( \frac{1}{2} x = \sqrt{25^2 - 3^2} = 4 \), so \( x = 8 \).
7. Given the quadrilateral $ABCD$ with sides $AB = 16$, $BC = 24$, $CD = 42$, $DA = 38$. Find the angle between diagonals.

**Solution.** Let $AC \cap BD = X$ and $\angle AXB = \alpha$. Using law of cosines

\[
\begin{align*}
256 &= AB^2 = AX^2 + BX^2 - 2 \cdot AX \cdot BX \cos \alpha \\
576 &= BC^2 = BX^2 + CX^2 - 2 \cdot BX \cdot CX \cos(180^\circ - \alpha) \\
1764 &= CD^2 = CX^2 + DX^2 - 2 \cdot CX \cdot DX \cos \alpha \\
1444 &= DA^2 = DX^2 + AX^2 - 2 \cdot DX \cdot AX \cos(180^\circ - \alpha)
\end{align*}
\]

So
\[
\begin{align*}
AX^2 + BX^2 + CX^2 + DX^2 &= 256 + 1764 - 2(AX \cdot BX + CX \cdot DX) \cos \alpha \\
AX^2 + BX^2 + CX^2 + DX^2 &= 576 + 1444 + 2(BX \cdot CX + DX \cdot AX) \cos \alpha.
\end{align*}
\]

And now good thing! $256+1764=576+1444=2020$, hence
\[-2(AX \cdot BX + CX \cdot DX) \cos \alpha = 2(BX \cdot CX + DX \cdot AX) \cos \alpha \Rightarrow \cos \alpha = 0 \Rightarrow \alpha = 90^\circ.
\]

8. Let us draw two squares $ABA'B'$ and $BCB''C'$ on sides $AB$ and $BC$ for $\triangle ABC$ facing outward. Let $E$ be the midpoint of $AA'$, $F$ be the midpoint of $CC'$, $G$ be the midpoint of $AC$ and $H$ be the midpoint of $A'C'$. Prove that $EHFG$ is a square.

**Solution 1 (rotation).** Let $T$ be rotation on $90^\circ$ clockwise.

\[
T(\overrightarrow{EG}) = T(\frac{1}{2}\overrightarrow{EA} + \frac{1}{2}\overrightarrow{AC}) = \frac{1}{2}T(\overrightarrow{EA} + \overrightarrow{AC}) = \frac{1}{2}\overrightarrow{EB} + \frac{1}{2}(\overrightarrow{BA'} + \overrightarrow{BC'}) = \frac{1}{2}\overrightarrow{EB} + \frac{1}{2}\overrightarrow{BH} = \overrightarrow{EH}.
\]

Similarly $T(\overrightarrow{FH}) = \overrightarrow{FG}$. Now we have $\triangle EGF = \triangle EHF$ by SSS ($EG = EH$, $GF = HF$, $EF$ shared), so in the quadrilateral $ECHF$ we have two angles are $90^\circ$ and two angles are equal, so all angles are $90^\circ$, and $EG = EH$, so $EGFH$ is square.

**Solution 2 (without vectors).** Consider quadrilateral $AA'C'C$. Points $E$, $G$, $H$, $F$ are midpoints of sides, so $EGHF$ is the parallelogram ($EG = HF = \frac{1}{2}A'C$, $EH = GF =$
Now we can do the same reasoning for \( \frac{1}{2}AC' \). So, we need to prove that diagonals \( A'C \) and \( CA' \) are equal and perpendicular. To prove it we can consider triangles \( BA'C \) and \( BAC' \) which are equal and corresponding sides are perpendicular.

9. In the triangle \( ABC \), let \( A_1 \) and \( A_2 \), \( B_1 \) and \( B_2 \), \( C_1 \) and \( C_2 \), are points on sides \( BC \), \( AC \), and \( AB \), respectively such that segments \( A_1B_2 \), \( B_1C_2 \) and \( C_1A_2 \) are concurrent. Let

\[
\frac{BA_1}{A_1C} = \alpha_1, \quad \frac{CB_1}{B_1A} = \beta_1, \quad \frac{AC_1}{C_1B} = \gamma_1, \quad \frac{CA_2}{A_2B} = \alpha_2, \quad \frac{AB_2}{B_2C} = \beta_2, \quad \frac{BC_2}{C_2A} = \gamma_2.
\]

Then

\[
\alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 + \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 1.
\]

**Solution.** Let \( X \) be the point of intersection \( A_1B_2 \), \( B_1C_2 \) and \( C_1B_2 \). Now consider \( X \) as a point on \( A_1B_2 \). We have

\[
A_{AXC} = \frac{XB_2}{A_1B_2} A_{AA_1C} = \frac{XB_2}{A_1B_2} \cdot \frac{A_1C}{BC} A_{ABC} \Leftrightarrow \frac{XB_2}{A_1B_2} = \frac{A_{AXC} BC}{A_{ABC} A_1C}.
\]

Similarly

\[
\frac{A_1X}{A_1B_2} = \frac{A_{BXC} AC}{A_{ABC} B_2C}.
\]

Take a sum and have (let \( A_{AXC} = x_B, A_{BXC} = x_A, A_{AXB} = x_C, A_{ABC} = x_A + x_B + x_C)\)

\[
x_B \frac{AC}{B_2C} + x_A \frac{AC}{B_2C} = x_A + x_B + x_C \Leftrightarrow x_B (1 + \alpha_1) + x_A (1 + \beta_2) = x_A + x_B + x_C \Leftrightarrow x_B \alpha_1 + x_A \beta_2 = x_C.
\]

Now we can do the same reasoning for \( B_1C_2 \) and \( C_1A_2 \) and have system of equations with nontrivial solution:

\[
\begin{align*}
x_B \alpha_1 + x_A \beta_2 &= x_C \\
x_C \beta_1 + x_B \gamma_2 &= x_A \\
x_A \gamma_1 + x_C \alpha_2 &= x_B
\end{align*} \Leftrightarrow \begin{align*}
x_C &= x_B \alpha_1 + x_A \beta_2 \\
(\beta_1 x_B \alpha_1 + x_A \beta_2) \beta_1 + x_B \gamma_2 &= x_A \\
x_A \gamma_1 + (x_B \alpha_1 + x_A \beta_2) \alpha_2 &= x_B
\end{align*} \Rightarrow \begin{align*}
x_B (\alpha_1 \beta_1 + \gamma_1) + x_A (\beta_1 \beta_2 - 1) = 0 \\
x_A (\gamma_1 + \alpha_2 \beta_2) + x_B (\alpha_1 \alpha_2 - 1) = 0.
\end{align*}
\]

To have nontrivial solution we should have

\[
\begin{align*}
\frac{\beta_1 \beta_2 - 1}{\gamma_1 + \alpha_2 \beta_2} &= \frac{\alpha_1 \beta_1 + \gamma_2}{\alpha_1 \alpha_2 - 1} \Leftrightarrow (\beta_1 \beta_2 - 1)(\alpha_1 \alpha_2 - 1) - (\gamma_1 + \alpha_2 \beta_2)(\alpha_1 \beta_1 + \gamma_2) = 0 \Leftrightarrow \\
\alpha_1 \beta_1 \gamma_1 + \alpha_2 \beta_2 \gamma_2 + \alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2 &= 1.
\end{align*}
\]