Department of Mathematics, University of Wisconsin-Madison

Math 431 — Midterm Exam 2 — Solutions — Spring 2025

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INSTRUCTIONS:

Time: **90 minutes**

- This exam contains 7 questions some with multiple parts, 12 pages (including the cover) for the total of 83 points. Read the problems carefully and budget your time wisely.
- You are allowed a single sheet of hand-written notes.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others.
- You **do not** need to simplify binomial coefficients or factorials, except when you are asked to do it.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded. Justify your steps.
- Remember, expectation and variance can be either finite, infinite or undefined.
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further.

Question:	1	2	3	4	5	6	7	Total
Points:	16	15	8	6	12	18	8	83
Score:								

1. (16 points) **Short problems. (4 points each)** In the following problems (but only here!) you will not need to justify your answer.

(a) Suppose that a random variable X has mean 2 and variance 3. Find numbers a, b so that $aX + b$ has mean 1 and variance 12.

Solution: We have seen in class that for a, b real numbers and a random variable X we have

$$E[aX + b] = aE[X] + b, \qquad \mathsf{Var}(aX + b) = a^2\mathsf{Var}(X).$$

Hence if a, b satisfy the requirements of the problem, then we have

$$1 = E[aX + b] = aE[X] + b = 2a + b$$

and

$$12 = \mathsf{Var}(aX + b) = a^2\mathsf{Var}(X) = 3a^2.$$

The last equation gives $a = \pm 2$, from which, using the first equation $2a + b = 1$, we get the possible solutions $a = 2, b = -3$ and $a = -2, b = 5$. (Either one would be an acceptable solution.)

(b) In humans (and many other organisms), genes come in pairs. Consider a gene of interest, which comes in two types (alleles): type a and type A. The genotype of a person for that gene is the types of the two genes in the pair: AA, Aa, or aa (aA is equivalent to Aa). According to the Hardy–Weinberg law, for a population in equilibrium, the frequencies of AA, Aa, aa will be p^2, $2p(1-p)$, $(1-p)^2$, respectively, for some p with $0 < p < 1$. Suppose that the Hardy–Weinberg law holds, and that n people are drawn randomly from the population, independently. Let X_1, X_2, X_3 be the number of people in the sample with genotypes AA, Aa, aa, respectively. What is the joint PMF of X_1, X_2, X_3?

Solution:

$$(X_1, X_2, X_3) \sim Multinomial(n, 3, p^2, 2p(1-p), (1-p)^2),$$

so

$$p_{X_1,X_2,X_3}(k_1, k_2, k_3) = \frac{n!}{k_1!k_2!k_3!} \left(p^2\right)^{k_1} \left(2p(1-p)\right)^{k_2} \left((1-p)^2\right)^{k_3}$$

when k_i are non-negative integers satisfying $k_1 + k_2 + k_3 = n$, and 0 otherwise.

(c) Suppose the duration of a phone call is given by an exponential distribution with an expected length of 15 minutes. Determine the conditional probability that a call lasts more than 20 minutes, given that it has already lasted for 10 minutes.(You don't have to simplify the answer.)

Solution: Denote by T the length of phone call. The expected value of an $\mathrm{Exp}(\lambda)$ is $1/\lambda$, hence $\lambda = 1/15$. By the memoryless property of the exponential distribution

$$P(T > 20|T > 10) = P(T > 10) = 1 - (1 - e^{-\lambda \cdot 10}) = e^{-\frac{10}{15}} = e^{-2/3}.$$

(d) Let X be a random variable with moment generating function

$$M(t) = \begin{cases} \dfrac{1}{\sqrt{1-2t}}, & |t| < \frac{1}{2}. \\ \infty, & \text{otherwise.} \end{cases}$$

Find the expectation $E[X]$, and the variance $\mathrm{Var}[X]$.

Solution:

$$E[X] = M'(0) = -\frac{1}{2}(1-2t)^{-3/2}(-2)\Big|_{t=0} = 1.$$

$$E[X^2] = M''(0) = (M')'(0) = ((1-2t)^{-3/2})'\Big|_{t=0} = -\frac{3}{2}(1-2t)^{-5/2}(-2)\Big|_{t=0} = 3.$$

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = 3 - 1^2 = 2.$$

2. Let X and Y be discrete random variables defined on the same sample space. The table below gives the values of the joint probability mass function of X, Y, for example, $p_{X,Y}(1,-1) = P(X = 1, Y = -1) = 1/8$.

$X \backslash Y$	-1	0	1
1	$\frac{1}{8}$	0	$\frac{2}{8}$
2	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{2}{8}$

(a) (4 points) Find the marginal probability mass functions of X and Y.

Solution: To find the marginal probabilities of X and Y, we need to sum the values of the joint pmf over rows and columns:

$$P(X = 1) = \frac{1}{8} + 0 + \frac{2}{8} = \frac{3}{8}$$
$$P(X = 2) = \frac{1}{8} + \frac{2}{8} + \frac{2}{8} = \frac{5}{8}$$

and

$$P(Y = -1) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8}$$
$$P(Y = 0) = 0 + \frac{2}{8} = \frac{2}{8}$$
$$P(Y = 1) = \frac{2}{8} + \frac{2}{8} = \frac{4}{8}$$

(b) (4 points) Are X and Y independent? Give a rigorous explanation.

Solution: X and Y are not independent. Indeed, if they were independent we would have $P(X = k_1, Y = k_2) = P(X = k_1)P(Y = k_2)$ for all $k_1 \in \{1, 2\}$ and all $k_2 \in \{-1, 0, 1\}$, but we have

$$P(X = 1, Y = 0) = 0 \qquad \text{and} \qquad P(X = 1)P(Y = 0) = \frac{3}{8} \cdot \frac{2}{8} > 0.$$

Hence, $P(X = 1, Y = 0) \neq P(X = 1)P(Y = 0)$.

(c) (4 points) Find $\mathrm{Var}(Y)$.

Solution: We have

$$E[Y] = (-1)\cdot P(Y=-1) + 0\cdot P(Y=0) + 1\cdot P(Y=1) = -\frac{2}{8} + \frac{4}{8} = \frac{1}{4}$$
$$E[Y^2] = (-1)^2\cdot P(Y=-1) + 0^2\cdot P(Y=0) + 1^2\cdot P(Y=1) = \frac{2}{8} + \frac{4}{8} = \frac{3}{4}$$

Hence,

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = \frac{3}{4} - \frac{1}{16} = \frac{11}{16}.$$

(d) (3 points) Find $P(Y < X)$.

Solution: We have

$$P(Y < X) = 1 - P(Y \geq X) = 1 - P(1,1) = 1 - \frac{2}{8} = \frac{3}{4}.$$

3. (8 points) Let $X \sim \text{Unif}\,[0,5]$. Find the probability density function of the random variable $Y = X^2$.

Solution: We will first find the cumulative distribution function (cdf) $F_Y(t)$ of Y.

The range of X is $[0,5]$, so the range of $Y = X^2$ is $[0,25]$.

Then we have, using that $X \sim \text{Unif}\,[0,5]$ and so $P(X \in [0,a]) = \frac{a}{5}$ for any $a \in [0,5]$:

$$F_Y(t) = P(X^2 \leq t,\ X \in [0,5]) = 0, \qquad \text{if } t < 0;$$

$$F_Y(t) = P(X^2 \leq t,\ X \in [0,5]) = P(X \in [0,\sqrt{t}]) = \frac{\sqrt{t}}{5}, \qquad \text{if } 0 \leq t < 25;$$

$$F_Y(t) = P(X^2 \leq t,\ X \in [0,5]) = 1, \qquad \text{if } t > 25.$$

Note that $F_Y(t)$ is continuous (we need to check that only at the points $t = 0$ and $t = 25$), and $F'(t)$ exists everywhere except these two points. Then the probability density function $f_Y(t)$ of Y is obtained by differentiating $F_Y(t)$, except the points $t = 0$ and $t = 25$ where we define $f_Y(t)$ arbitrarily. Then we have

$$f_Y(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1}{10\sqrt{t}}, & 0 < t < 25 \\ 0, & t \geq 25. \end{cases}$$

4. (6 points) The owner of a certain website is studying the distribution of the number of visitors to the site. Every day, a million $(1,000,000)$ people independently decide whether to visit the site, with probability $p = 2 \cdot 10^{-6}$ of visiting. Give a good approximation for the probability of getting at least three visitors on a particular day.

Solution: Let $X \sim \text{Bin}(n, p)$ be the number of visitors, where $n = 10^6$. Since n is large, p is small, and np^2 is very small (< 0.1), Poisson(2) is a good approximation. This gives

$$P(X \geq 3) = 1 - P(X < 3) \approx 1 - e^{-2} - e^{-2} \cdot 2 - e^{-2} \cdot \frac{2^2}{2!} = 1 - 5e^{-2} \approx 0.3233,$$

5. Let random variables X and Y have joint pdf

$$\begin{cases} A(x,y) & \text{for } 0 \le x \le 4, 0 \le y \le 3, \\ 0 & \text{otherwise} \end{cases}$$

Your integrals in the answers in this problem should be of the form $\int_a^b \int_c^d \ldots$ with appropriate limits of integration a, b, c, d and an explicit function inside the integral.

(a) (4 points) Which conditions do we need to check to be sure that this is actual joint pdf?

Solution: $\int_0^4 \int_0^3 A(x,y)dydx = 1$ and $A(x,y) \ge 0$ for any $0 \le x \le 4$, $0 \le y \le 3$.

(b) (4 points) Write an integral expression for the $E[X^2Y]$.

Solution:

$$E[X^2Y] = \int_0^4 \int_0^3 x^2 y A(x,y) dydx.$$

(c) (4 points) Write an integral expression for the $P(X+Y<2)$.

Solution:

$$P(X+Y<2) = \int_0^2 \int_0^{2-x} A(x,y)dydx$$

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6. Suppose random variable X has density function

$$f_X(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

(a) (6 points) Find $P(2 \leq X \leq 3)$

Solution:

$$P(2 \leq X \leq 3) = \int_2^3 \frac{2}{x^3} dx = -\frac{1}{x^2}\Big|_2^3 = -\frac{1}{9} + \frac{1}{4} = \frac{5}{36}.$$

(b) (6 points) Find $E[X]$

Solution:

$$E[X] = \int_1^\infty x \frac{2}{x^3} dx = \int_1^\infty \frac{2}{x^2} dx = -\frac{2}{x}\Big|_1^\infty = 2.$$

(c) (6 points) Find $\text{Var}[X]$

Solution:

$$E[X^2] = \int_1^\infty x^2 \frac{2}{x^3} dx = \int_1^\infty \frac{2}{x} dx.$$

Integral diverges, so $E[X^2] = +\infty$ and $\text{Var}(X) = +\infty$.

7. (8 points) Our ice machines produce ice cubes, 3 or 4 ice cubes every second with equal probabilities. Approximate the probability that that in 1 minutes and 40 seconds we will have at least 360 ice cubes.

Solution: Let $T_i = 3 + X_i$ be the number of cubes produced in the ith second, $X_i \sim Ber(\frac{1}{2})$, and let $S_{100} = 300 + X_1 + \ldots + X_{100}$ be the total number of cubes produced during 100 seconds. Note that $S_{100} \sim 300 + Bin(100, \frac{1}{2})$, $100 \cdot \frac{1}{2} \cdot \frac{1}{2} > 10$, so we can use Normal approximation

$$P(S_{100} \geq 360) = P(X_1 + \ldots + X_{100} \geq 60) = P\left(\frac{X_1 + \ldots + X_{100} - 50}{\sqrt{100 \cdot \frac{1}{2} \cdot \frac{1}{2}}} \geq \frac{60 - 50}{5}\right)$$

$$\approx 1 - \Phi(2) = 0.0228.$$

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Table of values for $\Phi(x)$, the CDF of a standard normal random variable

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

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