Department of Mathematics, University of Wisconsin-Madison

Math 431 — Midterm Exam 2 — Spring 2025

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INSTRUCTIONS:

Time: **90 minutes**

- This exam contains 7 questions some with multiple parts, 12 pages (including the cover) for the total of 83 points. Read the problems carefully and budget your time wisely.
- You are allowed a single sheet of hand-written notes.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others.
- You **do not** need to simplify binomial coefficients or factorials, except when you are asked to do it.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded. Justify your steps.
- Remember, expectation and variance can be either finite, infinite or undefined.
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further.

Question:	1	2	3	4	5	6	7	Total
Points:	16	15	8	6	12	18	8	83
Score:								

1. (16 points) **Short problems. (4 points each)** In the following problems (but only here!) you will not need to justify your answer.

   (a) Suppose that a random variable X has mean 2 and variance 3. Find numbers a, b so that $aX + b$ has mean 1 and variance 12.

   (b) In humans (and many other organisms), genes come in pairs. Consider a gene of interest, which comes in two types (alleles): type a and type A. The genotype of a person for that gene is the types of the two genes in the pair: AA, Aa, or aa (aA is equivalent to Aa). According to the Hardy–Weinberg law, for a population in equilibrium, the frequencies of AA, Aa, aa will be p^2, $2p(1-p)$, $(1-p)^2$, respectively, for some p with $0 < p < 1$. Suppose that the Hardy–Weinberg law holds, and that n people are drawn randomly from the population, independently. Let X_1, X_2, X_3 be the number of people in the sample with genotypes AA, Aa, aa, respectively. What is the joint PMF of X_1, X_2, X_3?

(c) Suppose the duration of a phone call is given by an exponential distribution with an expected length of 15 minutes. Determine the conditional probability that a call lasts more than 20 minutes, given that it has already lasted for 10 minutes.(You don't have to simplify the answer.)

(d) Let X be a random variable with moment generating function

$$M(t) = \begin{cases} \dfrac{1}{\sqrt{1-2t}}, & |t| < \frac{1}{2}. \\ \infty, & \text{otherwise.} \end{cases}$$

Find the expectation $E[X]$, and the variance $\mathrm{Var}[X]$.

2. Let X and Y be discrete random variables defined on the same sample space. The table below gives the values of the joint probability mass function of X, Y, for example, $p_{X,Y}(1,-1) = P(X=1, Y=-1) = 1/8$.

		Y		
		-1	0	1
X	1	$\frac{1}{8}$	0	$\frac{2}{8}$
	2	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{2}{8}$

(a) (4 points) Find the marginal probability mass functions of X and Y.

(b) (4 points) Are X and Y independent? Give a rigorous explanation.

(c) (4 points) Find $\text{Var}(Y)$.

(d) (3 points) Find $P(Y < X)$.

3. (8 points) Let $X \sim \text{Unif}[0, 5]$. Find the probability density function of the random variable $Y = X^2$.

4. (6 points) The owner of a certain website is studying the distribution of the number of visitors to the site. Every day, a million $(1,000,000)$ people independently decide whether to visit the site, with probability $p = 2 \cdot 10^{-6}$ of visiting. Give a good approximation for the probability of getting at least three visitors on a particular day.

5. Let random variables X and Y have joint pdf

$$\begin{cases} A(x,y) & \text{for } 0 \le x \le 4, 0 \le y \le 3, \\ 0 & \text{otherwise} \end{cases}$$

Your integrals in the answers in this problem should be of the form $\int_a^b \int_c^d \dots$ with appropriate limits of integration a, b, c, d and an explicit function inside the integral.

(a) (4 points) Which conditions do we need to check to be sure that this is actual joint pdf?

(b) (4 points) Write an integral expression for the $E[X^2Y]$.

(c) (4 points) Write an integral expression for the $P(X + Y < 2)$.

6. Suppose random variable X has density function

$$f_X(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

(a) (6 points) Find $P(2 \leq X \leq 3)$

(b) (6 points) Find $E[X]$

(c) (6 points) Find $\text{Var}[X]$

7. (8 points) Our ice machines produce ice cubes, 3 or 4 ice cubes every second with equal probabilities. Approximate the probability that that in 1 minutes and 40 seconds we will have at least 360 ice cubes.

First Name: _____ Last Name: _____

Table of values for $\Phi(x)$, the CDF of a standard normal random variable

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM.
SCRATCH WORK WILL NOT BE GRADED