Department of Mathematics, University of Wisconsin-Madison

Math 431 — Midterm Exam 1 — Solutions — Spring 2025

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INSTRUCTIONS:

Time: **90 minutes**

- This exam contains 6 questions some with multiple parts, 10 pages (including the cover) for the total of 83 points. Read the problems carefully and budget your time wisely.
- You are allowed a single sheet of hand-written notes.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others.
- You **do not** need to simplify binomial coefficients or factorials, except when you are asked to do it.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded.
- Justify your steps.
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further.

Question:	1	2	3	4	5	6	Total
Points:	16	12	8	18	21	8	83
Score:							

1. (16 points) The following questions are **Answers Only**.

   (a) In a five-card hand drawn at random from a well-shuffled standard deck, find the probability that the hand has exactly three aces.

   Solution:
   $$\frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}.$$

   (b) Let X and Y be random variables taking only values 1 and 2. Give the precise conditions for X and Y to be independent.

   Solution:
   $$P(X = 2, Y = 2) = P(X = 2)P(Y = 2)$$
   $$P(X = 1, Y = 2) = P(X = 1)P(Y = 2)$$
   $$P(X = 2, Y = 1) = P(X = 2)P(Y = 1)$$
   $$P(X = 1, Y = 1) = P(X = 1)P(Y = 1)$$
   One of equalities is enough.

   (c) We sample two numbers without replacement with order from the set $\{1, 2, 3, 4, 5, 6\}$. Describe the sample space of the experiment.

   Solution: $\Omega = \{(s_1, s_2) : \; s_1, s_2 \in \{1, 2, 3, 4, 5, 6\}, \;\; s_1 \neq s_2\}$

   (d) Suppose that A and B are events with $P(A) = \frac{1}{2}$, $P(B) = \frac{2}{3}$, and $P(A \cap B) = \frac{1}{4}$. Find the probability of $A^c \cup B$.

   Solution: We will use the formula:
   $$P(A^c \cup B) = P(A^c) + P(B) - P(A^c \cap B).$$
   First we find $P(A^c) = 1 - P(A) = \frac{1}{2}$.
   Next we find $P(A^c \cap B)$. Since $B = (A \cap B) \cup (A^c \cap B)$ and the sets $A \cap B$ and $A^c \cap B$ are disjoint, then $P(A^c \cap B) = P(B) - P(A \cap B) = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$.
   Now we have
   $$P(A^c \cup B) = P(A^c) + P(B) - P(A^c \cap B) = \frac{1}{2} + \frac{2}{3} - \frac{5}{12} = \frac{3}{4}.$$

2. A certain candy company put golden tickets in the wrappers of 2% of their candy bars. If you find a golden ticket, you win a trip to their candy factory. Assume that every candy bar is equally likely to have a golden ticket.

   (a) (6 points) Suppose you bought 15 candy bars. What is the probability you find exactly 3 tickets?

   Solution: Let the random variable X be the number of tickets in 15 candy bars. Then X counts the number of successes in 15 independent trials with success probability $p = \frac{2}{100} = \frac{1}{50}$. Then $X \sim \mathrm{Bin}(15, \frac{1}{50})$, and the probability you find exactly 3 tickets is

   $$P(X = 3) = \binom{15}{3} \left(\frac{1}{50}\right)^3 \left(\frac{49}{50}\right)^{12}.$$

   (b) (6 points) Suppose you buy candy bars until you find your first ticket. What is the probability that you must buy more than 50?

   Solution: You must buy more than 50 candy bars to find your first ticket only in the case when there is no tickets in the first 50 candy bars, which has the probability

   $$(1 - p)^{50} = \left(\frac{49}{50}\right)^{50}.$$

3. (8 points) A woman is pregnant with twin boys. Twins may be either identical or fraternal. Suppose that 1/3 of twins born are identical, that identical twins have a 50% chance of being both boys and a 50% chance of being both girls, and that for fraternal twins each twin independently has a 50% chance of being a boy and a 50% chance of being a girl. Given the above information, what is the probability that the woman's twins are identical?

Solution: By Bayes' rule,

$$P(\text{identical}|\text{B and B}) = \frac{P(\text{B and B}|\text{identical})P(\text{identical})}{P(\text{B and B})} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{2}{3}} = \frac{1}{2}.$$

4. A box contains 8 green balls, 6 red balls, and 4 yellow balls. Two balls are sampled uniformly at random without replacement.

   (a) (6 points) What is the probability that one ball is green and another is yellow? (First write down your sample space.)

Solution: The total number of balls is $8+6+4=18$. The number of ways to choose two balls out of 18 without replacement is $\binom{18}{2}$. Similarly, the number of ways to choose 1 green ball out of 8 and one yellow ball out of 4 are $\binom{8}{1}$ and $\binom{4}{1}$ respectively. Then

$$P(1\text{ green, } 1\text{ yellow}) = \frac{\binom{8}{1}\binom{4}{1}}{\binom{18}{2}} = \frac{32}{153}.$$

   (b) (6 points) What is the probability that at least one of the balls is red?

Solution: The number of pairs in which there are no red balls is $\binom{12}{2}$, where 12=8+4 is the total number of green and yellow balls. Then

$$P(\text{at least 1 red ball}) = 1 - P(\text{no red balls}) = 1 - \frac{\binom{12}{2}}{\binom{18}{2}} = \frac{87}{153} = \frac{29}{51}.$$

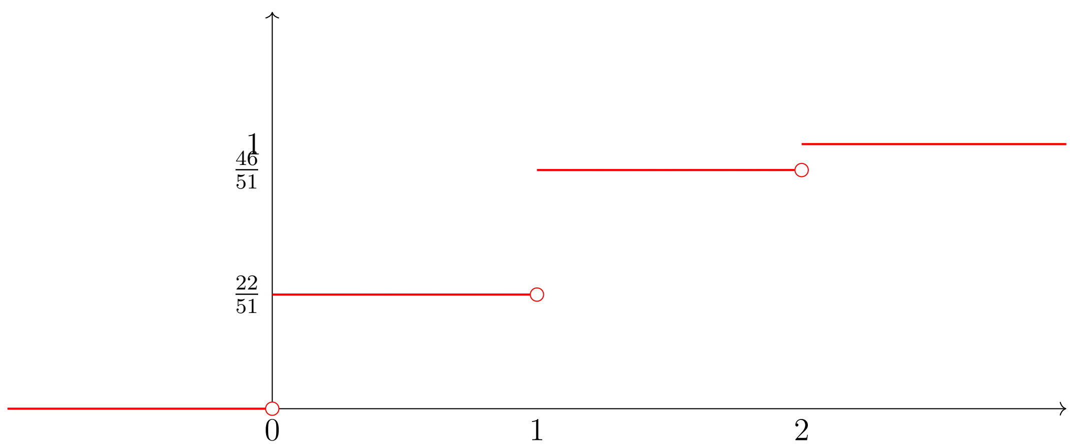
(c) (6 points) Let X be the number of red balls in the sample. Find CDF of X and draw the graph of this function.

Solution: X has the hypergeometric distribution with parameters $(18, 6, 2)$. Then the range of X is $\{0, 1, 2\}$, and

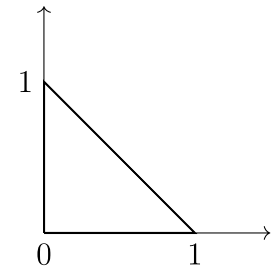
$$P(X=0)=\frac{\binom{6}{0}\binom{12}{2}}{\binom{18}{2}}=\frac{22}{51};\ \ P(X=1)=\frac{\binom{6}{1}\binom{12}{1}}{\binom{18}{2}}=\frac{8}{17};\ \ P(X=2)=\frac{\binom{6}{2}\binom{12}{0}}{\binom{18}{2}}=\frac{5}{51}.$$

Then CDF of X, the function $F(s) = P(X \le s)$, is

$$F(s)=\begin{cases}0, & s<0;\\ \frac{22}{51}, & 0\le s<1;\\ \frac{46}{51}, & 1\le s<2;\\ 1, & s\ge 2.\end{cases}$$



5. We choose a point (X, Y) uniformly from the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$. Let $Z = 2X - 1$.

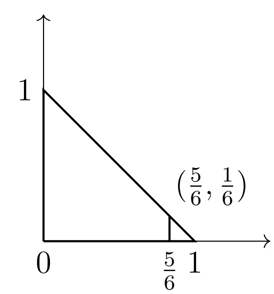


(a) (4 points) Find the range of the random variable Z.

Solution: Range of X is $[0, 1]$, and so the range of $Z = 2X - 1$ is $[-1, 1]$.

(b) (5 points) Compute $P(Z > \frac{2}{3})$.

Solution:



Denote by A the triangle defined in the problem. The region where $Z > \frac{2}{3}$ corresponds to the region where $2X - 1 > \frac{2}{3}$, which simplifies to $X > \frac{5}{6}$ within triangle A.
Thus, the region $\{X > \frac{5}{6}\}$ forms triangle B with vertices $\left(\frac{5}{6}, 0\right)$, $(1, 0)$, and $\left(\frac{5}{6}, \frac{1}{6}\right)$. This is a right triangle with two equal sides of length $\frac{1}{6}$. Then we have:

$$P\left(Z > \frac{2}{3}\right) = \frac{\text{area of triangle } B}{\text{area of triangle } A} = \frac{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{6}}{\frac{1}{2} \cdot 1 \cdot 1} = \frac{1}{36}.$$

(c) (6 points) Compute the cumulative distribution function of Z.

Solution: Since the range of Z is $[-1, 1]$, then cumulative distribution function $F(s)$ of Z satisfies $F(s) = 0$ for $s < -1$ and $F(s) = 1$ for $s \geq 1$.

Let $s \in [-1, 1)$. $F(s) = P(Z \leq s)$. We will first find $P(Z > s)$.

Similar to part (b) of this problem, the region where $Z > s$ corresponds to the region where $2X - 1 > s$, which simplifies to $X > \frac{s+1}{2}$ within triangle A. Note that $\frac{s+1}{2} \in [0, 1)$ since $s \in [-1, 1)$.

Thus, the region $\{Z > s\}$ forms triangle B_s with vertices $\left(\frac{s+1}{2}, 0\right)$, $(1, 0)$, and $\left(\frac{s+1}{2}, \frac{1-s}{2}\right)$, where $\frac{1-s}{2} = 1 - \frac{s+1}{2}$. This is a right triangle with two equal sides of length $\frac{1-s}{2}$. Then we have:

$$P(Z > s) = \frac{\text{area of triangle } B_s}{\text{area of triangle } A} = \frac{\frac{1}{2} \cdot \frac{1-s}{2} \cdot \frac{1-s}{2}}{\frac{1}{2} \cdot 1 \cdot 1} = \frac{(1-s)^2}{4}.$$

From this

$$F(s) = P(Z \leq s) = 1 - P(Z > s) = 1 - \frac{(1-s)^2}{4}.$$

Then the cumulative distribution function of Z is:

$$F(s) = \begin{cases} 0, & s < -1; \\ 1 - \frac{(1-s)^2}{4}, & -1 \leq s < 1; \\ 1, & s \geq 1. \end{cases}$$

(d) (6 points) Find the probability density function of Z.

Solution: Cumulative distribution function $F(s)$ is differentiable everywhere except $s = -1$ and $s = 1$. Then the probability density function $f(s)$ of Z is $f(s) = F'(s)$ at every point except $s = -1$ and $s = 1$, and defined arbitrarily at these two points, so

$$f(s) = \begin{cases} 0, & s < -1; \\ \frac{1-s}{2}, & -1 \leq s < 1; \\ 0, & s \geq 1. \end{cases}$$

6. (8 points) Professor May B. Right often has her facts wrong, and answers each of her students' questions incorrectly with probability 1/4, independent of other questions. In each lecture, May is asked 0, 1, or 2 questions with equal probability 1/3. Find the probability that she gave at least one wrong answer during one lecture.

Solution: Let N, be the number of questions that professor was asked and A be the event that she gave at least one wrong answer during one lecture. We know that $\{N=0\}$, $\{N=1\}$, $\{N=2\}$ is a partition, so

$$P(A) = P(N=0)P(A|N=0) + P(N=1)P(A|N=1) + P(N=2)P(A|N=2)$$

We know that $P(N=0) = P(N=1) = P(N=2) = \frac{1}{3}$, $P(A|N=0) = 0$ (there is no questions to answer), $P(A|N=1) = \frac{1}{4}$ and

$$P(A|N=2) = 1 - P(\text{both questions were answered correctly}) = 1 - \left(\frac{3}{4}\right)^2,$$

so, combining together

$$P(A) = \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{7}{16} = \frac{11}{48}.$$

OR

We can compute $P(A^c)$, i.e the probability that all answers were correct:

$$\begin{aligned} P(A^c) &= P(N=0)P(A^c|N=0) + P(N=1)P(A^c|N=1) + P(N=2)P(A^c|N=2) \\ &= \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{16+12+9}{48} = \frac{37}{48}. \end{aligned}$$

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM.
SCRATCH WORK WILL NOT BE GRADED